CHAPTER 8

Brownian Motions

布朗運動式個在財務數學上應用得相當廣泛的隨機過程.

- 1828 R. Brown: 觀察花粉在水中的活動.
- 1905 A. Einstein: fist mathematical theory about Brownian motion
- 1906 M. von Smoluchowski: same model as Einstein.
- 1923 N. Wiener: putting Brownian motion into the measure-theoretic framework.

8.1. Scaled random walk

在介紹 Brownian motion 之前, 我們先看看 symmetric random walk.

Construction of a symmetric random walk.
重複丟擲一個公平的銅板, 此銅板正反面出現的機率相同, 亦即

$$\text{the probability of H (head)} = \text{the probability of T (tail)} = \tfrac{1}{2}.$$

The successive outcome of the toss $\omega = \omega_1\,\omega_2\,\omega_3\,\cdots\,\omega_n\,\cdots$, where ω_n is the outcome of the nth toss. The sample space Ω is given by

$$\Omega = \{\omega : \omega = \omega_1\,\omega_2\,\cdots, \quad \omega_i = \text{ H or T }\}$$

Let

$$X_n(\omega) = \begin{cases} 1, & \text{if } \omega_n = H \\ -1, & \text{if } \omega_n = T \end{cases}$$

and $(X_n)_{n\geq 1}$ is independent.

Definition 8.1. Define

$$\begin{aligned} M_0 &= 0, \\ M_k &= \sum_{i=1}^{k} X_i, \qquad k = 1, 2, 3, \cdots \end{aligned}$$

The process $(M_k)_{k\geq 0}$ is a symmetric random walk.

Proposition 8.2. *A random walk has independent increments, i.e., any* $0 = t_0 < t_1 < t_2 < \cdots < t_m = t \quad (t_i \in \mathbb{N})$, *the increments of the random walk*

$$M_{t_1}, M_{t_2} - M_{t_1}, M_{t_3} - M_{t_2}, \cdots, M_{t_m} - M_{t_{m-1}}$$

are independent.

這個證明並不難, 依其定義即可.

Remark 8.3. The random variable

$$M_{t_k} - M_{t_{k-1}} = \sum_{i=t_{k-1}+1}^{t_k} X_i$$

has expectation 0 and variance $t_k - t_{k-1}$

Proof. Since

$$\begin{aligned} \mathbb{E}[X_i] &= 0, \\ \mathrm{Var}(X_i) &= \mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2 = 1, \end{aligned}$$

we have

$$\begin{aligned} \mathbb{E}[M_{t_k} - M_{t_{k-1}}] &= \sum_{i=t_{k-1}+1}^{t_k} \mathbb{E}[X_i] = 0, \\ \mathrm{Var}(M_{t_k} - M_{t_{k-1}}) &= \sum_{i=t_{k-1}+1}^{t_k} \mathrm{Var}(X_i) = t_k - t_{k-1} \end{aligned}$$

due to Proposition 8.2. □

Theorem 8.4. *(M_k) is a martingale with respect to $(\mathcal{F}_k^X)$.*

PROOF. Since $(X_n)_{n\geq 1}$ is independent,

$$\mathbb{E}[M_k - M_{k-1}|\mathcal{F}_{k-1}^X] = \mathbb{E}[X_k|\mathcal{F}_{k-1}^X] = \mathbb{E}[X_k] = 0.$$

□

Definition 8.5. Fixed a positive integer n, define the scaled symmetric random walk,

$$W_t^{(n)} = \frac{1}{\sqrt{n}} M_{nt},$$

Provided nt is an integer. If $nt \notin \mathbb{N}$, define $W_t^{(n)}$ by linear interpolation, i.e.,

$$W_t^{(n)} = ([nt] + 1 - nt) W_{\frac{[nt]}{n}}^{(n)} + (nt - [nt]) W_{\frac{[nt]+1}{n}}^{(n)}.$$

這個定法看似不易理解, 電用圖形來看應該會好一點. 見 Figures 8.1, 8.1 及 8.1.

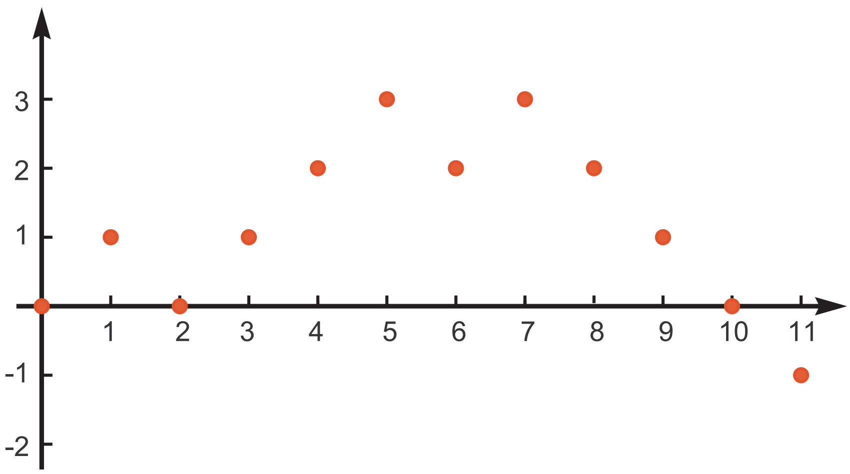

FIGURE 8.1. (M_k): 只有在整數點有值.

$(W_t^{(n)})$ 的想法則差不多. 上下跳動的大小為 $1/\sqrt{n}$, 跳動的頻率變成 $1/n$. 中間一樣用直線連起來.

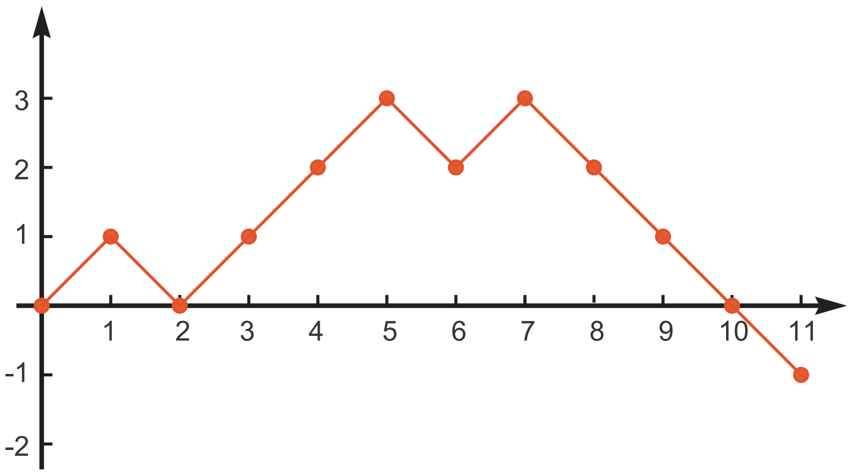

FIGURE 8.2. $(W_t^{(1)})$: 將 M_k 中間連起來即可.

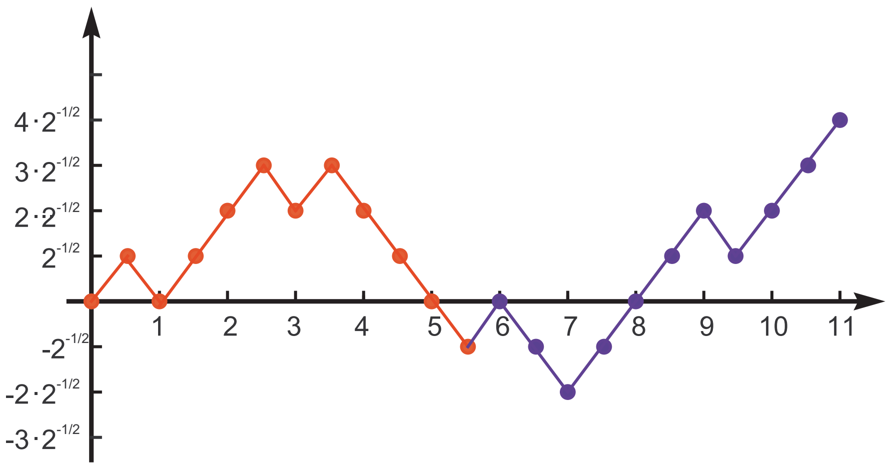

FIGURE 8.3. $(W_t^{(2)})$: 上下跳動的大小為 $1/\sqrt{2}$, 跳動的頻率變成 1/2. 中間一樣用直線連起來.

Proposition 8.6. *The scaled symmetric random walk has independent increments.*

PROOF. If $0 = t_0 < t_1 < t_2 < \cdots < t_m = t$ satisfy $nt_i \in \mathbb{N}$ for all i, then

$$W_{t_1}^{(n)}, W_{t_2}^{(n)} - W_{t_1}^{(n)}, W_{t_3}^{(n)} - W_{t_2}^{(n)}, \cdots, W_{t_m}^{(n)} - W_{t_{m-1}}^{(n)}$$

are independent. 至於一般的情況 ($nt_i \notin \mathbb{N}$) 就比較複雜, 必須分 case 討論. 我們在這裡就不講了. □

Theorem 8.7 (Central Limit Theorem). *Fixed $t \geq 0$. As $n \longrightarrow \infty$, the distribution of scalar symmetric random walk $(W_t^{(n)})$ evaluated at time t converges to $\mathcal{N}(0,t)$ in distribution.*

8.2. Brownian motions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 8.8. A stochastic process $W = (W_t)_{t\geq 0}$ is called a standard Brownian motion (BM) if

(i) $W_0 = 0$ $\mathbb{P}$-a.s.

(ii) (W_t) has independent increments, i.e., for $0 \leq t_1 \leq t_2 \leq \cdots \leq t_m$,

$$W_{t_1}, W_{t_2} - W_{t_1}, W_{t_3} - W_{t_2}, ..., W_{t_m} - W_{t_{m-1}}$$

are independent

(iii) For $0 \leq s < t$, $W_t - W_s \sim \mathcal{N}(0, t-s)$.

Remark 8.9. For all $t \geq 0$, $W_t \sim \mathcal{N}(0,t)$.

Remark 8.10. Difference between Brownian motion (W_t) and scaled symmetric random walk $(W_t^{(n)})$.

(1) The scaled random walk has a natural time step $1/n$ and is linear between these time steps.

(2) The scaled random walk $(W_t^{(n)})$ is only approximated normal for each t, but Brownian motion is exactly normal.

Lemma 8.11. *For $0 \leq s \leq t$, the covariance of W_s and W_t is s. Explicitly,*

$$\mathbb{E}[W_s W_t] = s \wedge t.$$

PROOF. Since $\mathbb{E}[W_s] = \mathbb{E}[W_t] = 0$, the covariance of W_s and W_t is given by

$$\begin{aligned}
\mathbb{E}[W_s W_t] &= \mathbb{E}[W_s(W_t - W_s + W_s)] = \mathbb{E}[W_s(W_t - W_s)] + \mathbb{E}[W_s^2] \\
&= \mathbb{E}[W_s]\mathbb{E}[W_t - W_s] + \mathbb{E}[W_s^2] = 0 + s = s
\end{aligned}$$

due to the independent increments of Brownian motion. ☐

Proposition 8.12. *The moment generating function of Brownian motion (for the m-dimensional random vector $(W_{t_1}, W_{t_2}, ..., W_{t_m})$) is given by*

$$\begin{aligned}
& \mathbb{E}\left[\exp\left(u_1 W_{t_1} + u_2 W_{t_2} + \cdots + u_m W_{t_m}\right)\right] \\
= \; & \exp\left(\frac{1}{2}(u_1 + u_2 + \cdots + u_m)^2 t_1 + \frac{1}{2}(u_2 + u_3 + \cdots + u_m)^2 (t_2 - t_1)\right. \\
& \left. + \cdots + \frac{1}{2}(u_{m-1} + u_m)^2 (t_{m-1} - t_{m-2}) + \frac{1}{2}\, u_m^2 (t_m - t_{m-1})\right)
\end{aligned}$$

PROOF. We prove here only the case $m = 3$. For $0 \le t_1 < t_2 < t_3$, due to the independence of W_{t_1}, $W_{t_2} - W_{t_1}$, $W_{t_3} - W_{t_2}$, we have

$$\begin{aligned}
& \mathbb{E}\left[\exp\left(u_1 W_{t_1} + u_2 W_{t_2} + u_3 W_{t_3}\right)\right] \\
= \; & \mathbb{E}\left[\exp\left(u_3(W_{t_3} - W_{t_2}) + (u_2+u_3)(W_{t_2} - W_{t_1}) + (u_1+u_2+u_3)w_{t_1}\right)\right] \\
= \; & \mathbb{E}\left[\exp\left(u_3(W_{t_3} - W_{t_2})\right)\right] \cdot \mathbb{E}\left[\exp\left((u_2+u_3)(W_{t_2} - W_{t_1})\right)\right] \cdot \mathbb{E}\left[\exp\left((u_1+u_2+u_3)w_{t_1}\right)\right] \\
= \; & \exp\left(\frac{1}{2}\, u_3^2(t_3 - t_2)\right) \exp\left(\frac{1}{2}\,(u_2+u_3)^2(t_2 - t_1)\right) \exp\left(\frac{1}{2}\,(u_1+u_2+u_3)^2 t_1\right) \\
= \; & \exp\left(\frac{1}{2}\,(u_1+u_2+u_3)^2 t_1 + \frac{1}{2}(u_2+u_3)^2(t_2 - t_1) + \frac{1}{2}u_3^2(t_3 - t_2)\right).
\end{aligned}$$

□

Definition 8.13. A filtration for Brownian motion (or Brownian filtration) is a collection of σ-algebra $\mathcal{F}_t$, $t \ge 0$, satisfying

(i) (information accumulates) $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \le t$;

(ii) (adaptivity) For each $t \ge 0$, W_t is $\mathcal{F}_t$-measurable;

(iii) (independent of future increment) For $0 \le s \le t$, $W_t - W_s$ is independent of $\mathcal{F}_s$.

Example 8.14. $\mathbb{F}^W = (\mathcal{F}_t^W)$ is a Brownian filtration.

Theorem 8.15. *Brownian motion is a martingale.*

PROOF. For $0 \le s < t$,

$$\begin{aligned}
\mathbb{E}[W_t|\mathcal{F}_s] &= \mathbb{E}[W_t - W_s|\mathcal{F}_s] + \mathbb{E}[W_s|\mathcal{F}_s] = \mathbb{E}[W_t - W_s] + W_s \\
&= 0 + W_s = W_s.
\end{aligned}$$

□

Exercise

(1) Let (W_t) be a one-dimensional standard Brownian motion. Check whether the following processes (X_t) are martingales with respect to $(\mathcal{F}_t)$:

(a) $X_t = W_t + 4t$.

(b) $X_t = W_t^2$.

(c) $X_t = t^2 W_t - 2\int_0^t sW_s\,ds$.

(d) $X_t = W_t^{(1)} W_t^{(2)}$, where $(W_t^{(1)})$ and $(W_t^{(2)})$ are two independent Brownian motions.

(2) Let (W_t) be a 1-dimensional standard Brownian motion.

(a) For fixed $t_0 \geq 0$, prove that

$$\bar{W}_t := W_{t_0+t} - W_{t_0}, \qquad t \geq 0$$

is a Brownian motion.

(b) Let c be a constant, prove that

$$\hat{W}_t := \frac{1}{c} W_{c^2 t}$$

is also a Brownian motion.

(c) Let σ be a constant. Show that

$$E\left[\exp\left(\sigma(W_t - W_s)\right)\right] = \exp\left(\frac{1}{2}\sigma^2(t-s)\right)$$

for $0 \leq s < t$.

8.3. The Brownian sample path

Theorem 8.16. (1) *There is a continuous version of Brownian motion.*[1]

(2) *For almost every $\omega \in \Omega$, the Brownian sample path $W_\cdot(\omega)$ is nowhere differentiable.*

(3) *For almost every $\omega \in \Omega$, the Brownian sample path $W_\cdot(\omega)$ is monotone in no interval.*

(4) *For almost every $\omega \in \Omega$, the set of points of local maximum for the Brownian sample path $W_\cdot(\omega)$ is countable and dense in $[0, \infty)$, and all local maxima are strict.*

Theorem 8.17 (Law of Iterated logarithm). *For almost every $\omega \in \Omega$, we have*

(1) $$\limsup_{t \downarrow 0} \frac{W_t(\omega)}{\sqrt{2t \log\log(1/t)}} = 1,$$
$$\liminf_{t \downarrow 0} \frac{W_t(\omega)}{\sqrt{2t \log\log(1/t)}} = -1.$$

(2) $$\limsup_{t \to \infty} \frac{W_t(\omega)}{\sqrt{2t \log\log(1/t)}} = 1,$$
$$\liminf_{t \to \infty} \frac{W_t(\omega)}{\sqrt{2t \log\log(1/t)}} = -1.$$

這個定理主要是在講述 Brownian motion 值的範圍. 當然, 這並不是說當 t 很大, 或 t 很小時, Brownian motion 會介於這兩個函數之間, 這主要是在顯示對所有的 sample path 來講, 在 t 很大, 或 t 很小時, Brownian motion 上升或下降的趨勢不會超過給定的這兩個函數. 亦即, 大多數的值會介於 $\pm\sqrt{2t \log\log t}$ 與 $\pm\sqrt{2t \log\log 1/t}$ 之間. $\sqrt{2t \log\log t}$ 與 $\sqrt{2t \log\log 1/t}$ 的圖形可見 Figure 8.3 及 Figure 8.3.

Theorem 8.18. *The quadratic variation of the standard Brownian motion is given by*

$$\langle W \rangle_t = t \qquad \mathbb{P}\text{ - } a.s. \qquad \text{for all } t \geq 0.$$

[1]所以以後我們會把 Brownian motion 想成有 continuous path!!! 這也是為何有些書上會在定義 Brownian motion 時直接定義其為連續的原因.

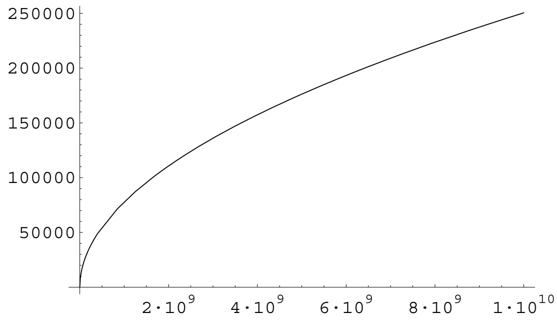

Figure 8.4. The graph of $\sqrt{2t\log\log t}$ when t is large

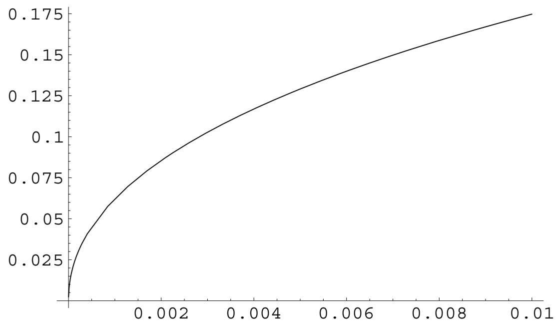

Figure 8.5. The graph of $\sqrt{2t\log\log\frac{1}{t}}$ when t is small

Proof. 在這裡我們用兩種不同的方法來求 quadratic variation.

Method 1: Let $\Pi = \{0 = t_0, t_1, t_2, \cdots, t_n\}$ be a partition of $[0,t]$. Define

$$\mathbb{Q}_\Pi = \sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2.$$

Then

$$\mathbb{E}[\mathbb{Q}_\Pi] = \sum_{i=0}^{n-1}\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2] = \sum_{i=0}^{n-1}(t_{i+1} - t_i) = t,$$

and

$$\begin{aligned}\mathbb{E}[(\mathbb{Q}_\Pi - t)^2] &= \mathbb{E}\left[\left(\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i)\right)^2\right] \\ &= \sum_{i=0}^{n-1}\mathbb{E}\left[\left((W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i)\right)^2\right] \\ &\quad +2\sum_{i<j}\mathbb{E}\left[\left((W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i)\right)\left((W_{t_{j+1}} - W_{t_j})^2 - (t_{j+1} - t_j)\right)\right].\end{aligned}$$

For $i < j$, since $W_{t_{i+1}} - W_{t_i}$ and $W_{t_{j+1}} - W_{t_j}$ are independent,

$$\begin{aligned}&\mathbb{E}\left[\left((W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i)\right)\left((W_{t_{j+1}} - W_{t_j})^2 - (t_{j+1} - t_j)\right)\right] \\ =\ & \mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})^2(W_{t_{j+1}} - W_{t_j})^2\right] - (t_{i+1} - t_i)\mathbb{E}[(W_{t_{j+1}} - W_{t_j})^2] \\ & -(t_{j+1} - t_j)\mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})^2\right] + (t_{i+1} - t_i)(t_{j+1} - t_j) \\ =\ & \mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})^2\right]\cdot\mathbb{E}\left[(W_{t_{j+1}} - W_{t_j})^2\right] - (t_{i+1} - t_i)(t_{j+1} - t_j) \\ & -(t_{j+1} - t_j)(t_{i+1} - t_i) + (t_{i+1} - t_i)(t_{j+1} - t_j) \\ =\ & (t_{i+1} - t_i)(t_{j+1} - t_j) - (t_{j+1} - t_j)(t_{i+1} - t_i) \\ =\ & 0.\end{aligned}$$

Thus,

$$\mathbb{E}[(\mathbb{Q}_\Pi - t)^2] = \sum_{i=0}^{n-1}\left(\mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})^4\right] - 2(t_{i+1} - t_i)\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2] + (t_{i+1} - t_i)^2\right).$$

Since

$$\mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})^4\right] = \frac{1}{\sqrt{2\pi(t_{i+1} - t_i)}}\int_{-\infty}^{\infty} x^4 e^{-\frac{x^2}{2(t_{i+1}-t_i)}}\,dx = 3(t_{i+1} - t_i)^2,$$

we get

$$\begin{aligned}\mathbb{E}[(\mathbb{Q}_\Pi - t)^2] &= \sum_{i=0}^{n-1}\big(3(t_{i+1}-t_i)^2 - 2(t_{i+1}-t_i)(t_{i+1}-t_i) + (t_{i+1}-t_i)^2\big)\\ &= 2\sum_{i=0}^{n-1}(t_{i+1}-t_i)^2 \le 2\sum_{i=0}^{n-1}\|\Pi\|(t_{i+1}-t_i) = 2t\|\Pi\| \longrightarrow 0,\end{aligned}$$

as $\|\Pi\| \longrightarrow 0$, where $\|\Pi\| = \max|t_{i+1}-t_i|$. Hence,

$$\lim_{\|\Pi\|\longrightarrow 0}\mathbb{E}[(\mathbb{Q}_\Pi - t)^2] = 0,$$

i.e.,

$$\lim_{\|\Pi\|\longrightarrow 0}\mathbb{Q}_\Pi = t \qquad \text{in } L^2.$$

而利用 subsequence convergence 的性質, 我們可以得到 almost everywhere convergence.

Method 2: Claim : $W_t^2 - t$ is a martingale.

For $0 \le s \le t$,

$$\begin{aligned}\mathbb{E}[W_t^2 - t|\mathcal{F}_s] &= \mathbb{E}[(W_t - W_s + W_s)^2|\mathcal{F}_s] - t\\ &= \mathbb{E}[(W_t - W_s)^2 + 2W_s(W_t - W_s) + W_s^2|\mathcal{F}_s] - t\\ &= \mathbb{E}[(W_t - W_s)^2|\mathcal{F}_s] + 2\mathbb{E}[W_s(W_t - W_s)|\mathcal{F}_s] + \mathbb{E}[W_s^2|\mathcal{F}_s] - t\\ &= \mathbb{E}[(W_t - W_s)^2] + 2W_s\mathbb{E}[W_t - W_s] + W_s^2 - t\\ &= t - s + 0 + W_s^2 - t = W_s^2 - s.\end{aligned}$$

Due to Doob-Meyer decomposition, we have $\langle W\rangle_t = t$. □

Remark 8.19. Let $\Pi = \{t_0, t_1, t_2, \cdots, t_n\}$ be a partition of $[0,t]$. Then

$$\lim_{\|\Pi\|\longrightarrow 0}\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 = t,$$

$$\lim_{\|\Pi\|\longrightarrow 0}\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i) = 0,$$

$$\lim_{\|\Pi\|\longrightarrow 0}\sum_{i=0}^{n-1}(t_{i+1}-t_i)^2=0.$$

We write informally

$$dW_t\cdot dW_t=dt,$$

$$dW_t\cdot dt=0,$$

$$dt\cdot dt=0.$$

8.4. Exponential martingales

一般股票價格並不會像 Brownian motion 一樣有正有負. 因此我們想找個函數 preserving Brownian motion 的 monotonicity, 又絶對是正的. 最簡單的想法便是 exponential function. 但這是由函數的性質得到的猜想, 我們有沒有辦法由財務的觀點得出 exponential function 呢? 首先先看看 binomial model. 如 Figure 8.4. 此為時間間隔為 1 的情況. 我們獻在觀察一下時間間隔為 $1/n$ 的狀況.

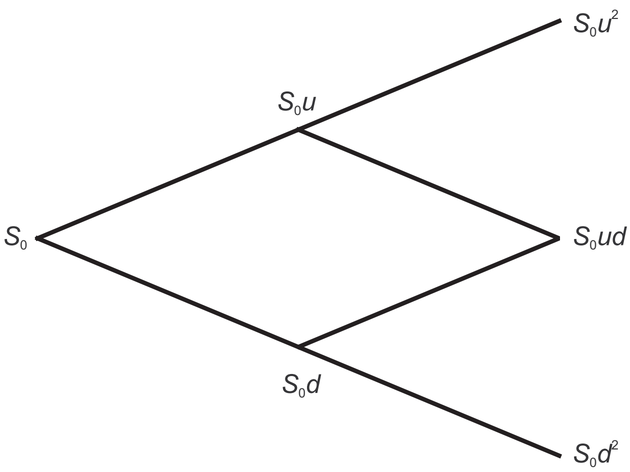

FIGURE 8.6. Binomial model

Suppose the interest rate $r=0$,

$$u_n=1+\frac{\sigma}{\sqrt{n}},\qquad d_n=1-\frac{\sigma}{\sqrt{n}},$$

where σ is a positive constant (called volatility).

Assume that $nt \in \mathbb{N}$, that means that up to time t, it is an nt-period model. The risk-neutral probability measure in the one-period model is given by

$$\tilde{p} = \frac{1-d_n}{u_n - d_n} = \frac{1-(1-\frac{\sigma}{\sqrt{n}})}{(1+\frac{\sigma}{\sqrt{n}})-(1-\frac{\sigma}{\sqrt{n}})} = \frac{1}{2}$$

$$\tilde{q} = \frac{u_n - 1}{u_n - d_n} = \frac{(1+\frac{\sigma}{\sqrt{n}})-1}{(1+\frac{\sigma}{\sqrt{n}})-(1-\frac{\sigma}{\sqrt{n}})} = \frac{1}{2}$$

Just like to toss a fair coin. This implies that we may regard the nt-period model as tossing a fair coin nt times.

Suppose

H_{nt} = the number of heads in the first nt coin tosses.

T_{nt} = the number of tails in the first nt coin tosses.

M_{nt} = the position of the 1-dimensional random walk. Clearly, we have

$$\begin{cases} H_{nt} + T_{nt} = nt, \\ H_{nt} - T_{nt} = M_{nt}. \end{cases}$$

Thus,

$$\begin{cases} H_{nt} = \dfrac{nt + M_{nt}}{2}, \\ T_{nt} = \dfrac{nt - M_{nt}}{2}. \end{cases}$$

This implies that the stock price at time t is given by

$$S_t^{(n)} = S_0\, u_n^{H_{nt}}\, d_n^{T_{nt}} = S_0 \left(1 + \frac{\sigma}{\sqrt{n}}\right)^{\frac{nt+M_{nt}}{2}} \left(1 - \frac{\sigma}{\sqrt{n}}\right)^{\frac{nt-M_{nt}}{2}}$$

Theorem 8.20. *As $n \longrightarrow \infty$, the distribution of $S_t^{(n)}$ converges to the distribution of*

$$S_t = S_0 \exp\left(\sigma W_t - \frac{\sigma^2 t}{2}\right)$$

where $W_t \sim \mathcal{N}(0, t)$

PROOF. Claim : the distribution of $\log S_t^{(n)}$ converges to the distribution of $\log S_t$, i.e.,

$$\log S_t^{(n)} \longrightarrow \log S_t \qquad \text{in distribution.}$$

$$\begin{aligned}
\log S_t^{(n)} &= \log S_0 + \frac{nt + M_{nt}}{2} \log(1 + \frac{\sigma}{\sqrt{n}}) + \frac{nt - M_{nt}}{2} \log(1 - \frac{\sigma}{\sqrt{n}}) \\
&= \log S_0 + \frac{nt + M_{nt}}{2} \left(\frac{\sigma}{\sqrt{n}} - \frac{\sigma^2}{2n} + O(n^{-\frac{3}{2}}) \right) \\
&\quad + \frac{nt - M_{nt}}{2} \left(-\frac{\sigma}{\sqrt{n}} - \frac{\sigma^2}{2n} + O(n^{-\frac{3}{2}}) \right) \\
&= \log S_0 + nt \left(-\frac{\sigma^2}{2n} + O(n^{-\frac{3}{2}}) \right) + M_{nt} \left(\frac{\sigma}{\sqrt{n}} + O(n^{-\frac{3}{2}}) \right) \\
&= \log S_0 - \frac{\sigma^2 t}{2} + O(n^{-\frac{1}{2}}) + \sigma W_t^{(n)} + O(n^{-1}) W_t^{(n)},
\end{aligned}$$

which converges in distribution to

$$\log S_0 + \sigma W_t - \frac{\sigma^2 t}{2} = \log S_t.$$

□

Definition 8.21. Let (W_t) be a Brownian motion with filtration $(\mathcal{F}_t)$, $\sigma \in \mathbb{R}$. The exponential martingale corresponding to σ is defined by

$$Z_t = \exp\left(\sigma W_t - \frac{\sigma^2}{2} t \right)$$

(A special case of geometric Brownian motion)

Theorem 8.22. *$(Z_t, \mathcal{F}_t)_{t \geq 0}$ is a martingale.*

Proof. For $0 \le s \le t$,

$$\begin{aligned}
\mathbb{E}[Z_t|\mathcal{F}_s] &= \mathbb{E}\left[\exp\left(\sigma W_t - \frac{\sigma^2 t}{2}\right)\Big|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[\exp(\sigma W_t)|\mathcal{F}_s\right] \cdot \exp\left(-\frac{\sigma^2 t}{2}\right) \\
&= \mathbb{E}[\exp\left(\sigma(W_t - W_s) + \sigma W_s\right)|\mathcal{F}_s] \cdot \exp\left(-\frac{\sigma^2 t}{2}\right) \\
&= \mathbb{E}[\exp\left(\sigma(W_t - W_s)\right)|\mathcal{F}_s] \cdot \exp\left(\sigma W_s\right) \cdot \exp\left(-\frac{\sigma^2 t}{2}\right).
\end{aligned}$$

Since

$$\mathbb{E}[\exp\left(\sigma(W_t - W_s)\right)|\mathcal{F}_s] = \mathbb{E}[\exp\left(\sigma(W_t - W_s)\right)] = \exp\left(\frac{1}{2}\sigma^2(t-s)\right),$$

we have

$$\begin{aligned}
\mathbb{E}[Z_t|\mathcal{F}_s] &= \exp\left(\frac{1}{2}\sigma^2(t-s)\right) \cdot \exp\left(\sigma W_s\right) \cdot \exp\left(-\frac{\sigma^2 t}{2}\right) \\
&= \exp\left(\sigma W_s - \frac{1}{2}\sigma^2 s\right) = Z_s.
\end{aligned}$$

□

8.5. d-dimensional Brownian motions

Definition 8.23. A d-dimensional stochastic process $B = (B_t)_{t\ge 0} = ((B_t^1, B_t^2, \cdots, B_t^d))_{t\ge 0}$ is called a d-dimensional Brownian motion if every $(B_t^i)_{t\ge 0}$ is a 1-dimensional Brownian motion and (B_t^1), (B_t^2), ..., (B_t^d) are independent.

Remark 8.24. A d-dimensional Brownian motion is a d-dimensional continuous martingale with cross variation.

$$\langle B^i, B^j\rangle_t = \delta_{ij} t\,, \qquad \text{for} \quad 1 \le i, j \le d.$$

Theorem 8.25 (Lévy Theorem). *Let $M = (M^1, M^2, \cdots, M^d)$ be a d-dimensional continuous local martingale with respect to $(\mathcal{F}_t)$ and $M_0 = 0$ $\mathbb{P}$-a.s. If*

$$\langle M^i, M^j\rangle_t = \delta_{ij} t\,, \qquad \text{for} \quad 1 \leq i, j \leq d, \tag{8.1}$$

then M is a d-dimensional Brownian motion.

Remark 8.26. The condition "continuity" is important. 例如 Poisson process 即為滿足 (8.1) 之 martingale, 但顯然 Poisson process 並不是個 martingale.

Theorem 8.27. *Let M be a real-valued continuous local martingale with respect to $(\mathcal{F}_t)$ and $M_0 = 0$ $\mathbb{P}$-a.s. with*

$$\lim_{t\longrightarrow\infty} \langle M\rangle_t = \infty.$$

For each $t \geq 0$, define the stopping time

$$\tau_t = \inf\{s : \langle M\rangle_s > t\}$$

Then $(M_{\tau_t})_{t\geq 0}$ is a Brownian motion with respect to $(\mathcal{F}_{\tau_t})$.

Remark 8.28. Let B be a d-dimensional Brownian motion starting from 0.

(1) If $d = 1$, $B_t(\omega)$ visits 0 infinite many times.

(2) If $d = 2$, $B_t(\omega)$ does not hit the origin after time 0, however it hits every ball with center at the origin.

(3) If $d \geq 3$, $|B_t(\omega)| \longrightarrow \infty$ as $t \to \infty$.

Exercise

(1) **Definition:** A Poisson process with intensity $\lambda > 0$ is an adapted, integer-valued càdlàg (right-continuous and left limit exists) process $N = (N_t, \mathcal{F}_t)_{t\geq 0}$ such that

(i) $N_0 = 0$, $\mathbb{P}$-a.s.;

(ii) $N_t - N_s$ is independent of $\mathcal{F}_s$, for $0 \leq s \leq t$;

(iii) $N_t - N_s$ is Poisson distributed with mean $\lambda(t-s)$.

Given a Poisson process N with intensity λ, define the compensated Poisson process

$$M_t = N_t - \lambda t.$$

(a) $(M_t, \mathcal{F}_t)$ is a martingale.

(b) Is the Poisson process $N = (N_t, \mathcal{F}_t)_{t\geq 0}$ a martingale, submartingale or supermartingale?

(c) Show that the quadratic variation of the compensated Poisson process M is given by $\langle M \rangle_t = \lambda t$.

(2) Let (W_t) be a d-dimensional Brownian motion starting at 0 and let $U \in \mathbb{R}^{d\times d}$ be a constant orthogonal matrix, i.e., $UU^T = I$. Prove that

$$\tilde{B}_t := UB_t$$

is also a Brownian motion.