UNIVERSITÉ DE NICE
SOPHIA ANTIPOLIS
U.F.R Sciences

ANNEE UNIVERSITAIRE 20__ / 20__ Session : ____

U.E. ____________________

Épreuve de ____________________

Note de l'épreuve

(1) Le candidat doit inscrire ici : ses noms, prénoms, lieu et date de naissance, puis rabattre suivant le pointillé le coin de la copie et le coller.

Il est interdit au candidat de signer sa copie ou d'y inscrire un signe quelconque pouvant en indiquer la provenance.

(1) Nom : ....
Prénoms : ....
Né(e) : ....
le : ....

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Nombre d'intercalaires ______ ________, le ____________ 20 ______

(1) We know that at time n, the wealth is W_n. The capital invested in the risky asset is $\hat{\lambda}_n W_n$ so that the numbers of shares that are detained is $\frac{1}{S_n}\hat{\lambda}_n W_n$.

We get $\frac{1}{S_n}\hat{\lambda}_n W_n D$ as a dividend.

Therefore: $W_{n+1}^{\hat{\lambda}_n} = \underbrace{\hat{\lambda}_n W_n}_{\text{capital in the risky asset}} \underbrace{\xi_{n+1}}_{\text{evolution of the capital}} + \underbrace{\frac{\hat{\lambda}_n W_n D}{S_n}}_{\text{Dividend}} + (1-\hat{\lambda}_n) W_n \underbrace{(1+r)}_{\text{evolution of the capital in the non risky asset}}$

Since $\hat{\lambda}_n \in [0,1]$ $W_{n+1}^{\hat{\lambda}_n} \geq 0$.

(2) (a) We write:

$$\sup_{\hat{\lambda} \in [0,1]} \mathbb{E}\left[U(W_1^{\hat{\lambda}})\right] = \sup_{\hat{\lambda} \in [0,1]} \mathbb{E}\left[\left(\hat{\lambda} x\left(\xi + \frac{D}{S}\right) + (1-\hat{\lambda}) x(1+r)\right)^q\right]$$

$$\left\{\underbrace{x^q}_{\geq 0} \mathbb{E}\left[\left(\hat{\lambda}\left(\xi + \frac{D}{S}\right) + (1-\hat{\lambda})(1+r)\right)^q\right]\right\}$$

$$= x^q \underbrace{\sup_{\hat{\lambda} \in [0,1]} \mathbb{E}\left[\left[\hat{\lambda}\left(\xi + \frac{D}{S}\right) + (1-\hat{\lambda})(1+r)\right]^q\right]}_{c(\lambda)}.$$

We notice that $c(\lambda) \geq 0$ since $\hat{\lambda} \in [0,1]$ in the $\mathbb{E}$. We even have $c(s) > 0$.

We check that c is non-increasing.

Take $0 \leq s < s'$. Since $\hat{\lambda}$ is taken in $[0,1]$ in the $\mathbb{E}$ that defines the reward function, it suffices to notice that

$$\forall \hat{\lambda} \in [0,1] \qquad \hat{\lambda} \cdot \frac{D}{s'} < \hat{\lambda} \cdot \frac{D}{s}$$

so that

$$\left[\hat{\lambda} \cdot \left(\zeta + \frac{D}{s'}\right) + (1-\hat{\lambda}) \cdot (1+r)\right]^q \leq \left[\hat{\lambda} \cdot \left(\zeta + \frac{D}{s}\right) + (1-\hat{\lambda}) \cdot (1+r)\right]^q$$

Taking the expectations and then the supremum over $\hat{\lambda}$, we complete the proof.

(b) We consider the function (the value of s being given)

$$J(\hat{\lambda}) = \mathbb{E}\left[\left(\hat{\lambda}\left(\zeta + \frac{D}{s}\right) + (1-\hat{\lambda})(1+r)\right)^q\right]$$

for $\hat{\lambda} \in [0,1]$

$$= p\left[\hat{\lambda}\left[u + \frac{D}{s} - (1+r)\right] + 1 + r\right]^q + (1-p)\left[\hat{\lambda}\left[d + \frac{D}{s} - (1+r)\right]\right]^q$$

We notice that J is twice differentiable in $\hat{a}$ on $[0,1]$. We compute J''. It reads:

$$J''(\hat{a}) = p\, q(q-1)\left[u + \frac{D}{s} - (1+r)\right]^2 \left[\hat{a}\left[u + \frac{D}{s} - (1+r)\right] + (1+r)\right]^{q-2}$$
$$+ (1-p)\, q(q-1)\left[d + \frac{D}{s} - (1+r)\right]^2 \left[\hat{a}\left[d + \frac{D}{s} - (1+r)\right] + (1+r)\right]^{q-2}.$$

We see that $p\,q(q-1) < 0$ $(1-p)\,q(q-1) < 0$.
Moreover, since $d < u$, we have

$$\left[u + \frac{D}{s} - (1+r)\right]^2 < 0 \text{ or } \left[d + \frac{D}{s} - (1+r)\right]^2 < 0.$$

Now:

$$\hat{a}\left[u + \frac{D}{s} - (1+r)\right] + 1+r = \hat{a}\left(u + \frac{D}{s}\right) + (1+r)(1-\hat{a}) \geq 0.$$

$$\hat{a}\left[d + \frac{D}{s} - (1+r)\right] + 1+r = \hat{a}\left(d + \frac{D}{s}\right) + (1+r)(1-\hat{a}) > 0.$$

Therefore $J''(\hat{a}) < 0$ $\forall\, \hat{a} \in [0,1]$.

By concavity the function J admits a unique maximum.

Of course the maximum does not depend on x since J does not depend on x.

3) (a) S_n may be $S_0 d^k u^{n-k}$ for k running along $\{0, \dots, n\}$. These are always > 0 values.

(b) By definition $U_N(x,s) = U(x)$.

(c) We expect that:

$$U_n(x,s) = \sup_{\hat{a} \in [0,1]} \mathbb{E}\left[U_{n+1}(W_{n+1}, S_{n+1})\right]$$

where in the sup: $\begin{cases} W_{n+1} = x\hat{a}\left(\xi + \frac{D}{s}\right) + (1+r)x(1-\hat{a}) \\ S_{n+1} = s\,\xi. \end{cases}$

(4) (a) We proceed by induction

Assume that $U_{n+1}(x,s) = c_n(s) x^q$

for any $x \geq 0$, $s > 0$ at some time $n+1 \leq N$.

Then, D.P.P says that

$$U_n(x,s) = \sup_{\hat{\alpha} \in [0,1]} \mathbb{E}\left[c_{n+1}^q(s\xi) \left\{ x\hat{\alpha}\left(\xi + \frac{D}{s}\right) + (1+r)(1-\hat{\alpha})x \right\}^q \right]$$

$$= x^q \sup_{\hat{\alpha} \in [0,1]} \mathbb{E}\left[c_{n+1}^q(s\xi) \left\{ \hat{\alpha}\left(\xi + \frac{D}{s}\right) + (1+r)(1-\hat{\alpha}) \right\}^q \right]$$

since $x^q \geq 0$

We get $c_n(s)$

If $c_{n+1}(s) > 0$ then $c_n(s) \geq 0$.

Assume that c_{n+1} is non-increasing. Then for any $\hat{\alpha} \in [0,1]$ we must have (since "everything" is non-negative) that the function

$$\mathbb{R}_+^* \ni s \mapsto c_{n+1}(s\xi) \left\{ \hat{\alpha}\left(\xi + \frac{D}{s}\right) + (1+r)(1-\hat{\alpha}) \right\}^q$$

is non-increasing whatever the realisation of ξ is. Proceeding as in question (2a), we deduce that c_n is non-increasing.

(b) The induction shows that $c_N = 1$ and

$$c_n(s) = \sup_{\hat{\alpha} \in [0,1]} \mathbb{E}\left[c_{n+1}(s\xi) \left\{ \hat{\alpha}\left(\xi + \frac{D}{s}\right) + (1+r)(1-\hat{\alpha}) \right\}^q \right].$$

(c) At time n, the financial agent has to optimize the above reward. Namely, he/she must take

$$\hat{\alpha}_n^*(s) = \sup_{\hat{\alpha} \in [0,1]} \mathbb{E}\left[c_{n+1}(s\xi) \left\{ \hat{\alpha}\left(\xi + \frac{D}{s}\right) + (1+r)(1-\hat{\alpha}) \right\}^q \right].$$

We proceed as in question (2b). We write:

$$\mathbb{E}\left[c_{n+1}(sZ)\left\{\hat{\alpha}\left(Z+\frac{D}{s}\right)+(1+r)(1-\hat{\alpha})\right\}^q\right]$$

$$= p\, c_{n+1}(su)\left\{\hat{\alpha}\left(u+\frac{D}{s}\right)+(1+r)(1-\hat{\alpha})\right\}^q$$

$$+ (1-p)\, c_{n+1}(sd)\left\{\hat{\alpha}\left(d+\frac{D}{s}\right)+(1+r)(1-\hat{\alpha})\right\}^q.$$

} We call it $J_n(\hat{\alpha})$

J_n is twice differentiable and we may prove that

$$J_n''(\hat{\alpha}) < 0 \quad \forall\, \hat{\alpha} \in [0,1]$$

by the same argument as in question (2b). We must only pay attention to the fact that, by induction, the coefficients $c_{n+1}(su)$ and $c_{n+1}(sd)$ must be > 0.

Conclusion. J_n is strictly concave and must have a unique maximizer on $[0,1]$. We denote it by $\alpha_n^*(s)$.

At time n, the financial agent proceed with the allocation by computing

$$\alpha_n^*(S_n)$$

The only needed information is the spot price S_n at time n.

Question (5)

(a) Starting from a wealth x at time n, the wealth at time $n+1$ reads:

$$x\hat{\alpha}\xi + x(1-\hat{\alpha})(1+r) \quad \text{(No dividend since } D=0\text{)}$$

When $\xi = u$, we get

$$x\hat{\alpha}u + x(1-\hat{\alpha})(1+r) = \underbrace{x\hat{\alpha}}_{\geq -\frac{x(1+r)}{u-(1+r)}}\underbrace{\{u-(1+r)\}}_{\geq 0} + x(1+r)$$

$$\geq -x(1+r) + x(1+r) = 0$$

Similarly when $\xi = d$, we get

$$x\hat{\alpha}d + x(1-\hat{\alpha})(1+r) = \underbrace{x\hat{\alpha}}_{\leq \frac{x(1+r)}{1+r-d}}\overbrace{\{d-(1+r)\}}^{\leq 0} + x(1+r)$$

$$\geq -x(1+r) + x(1+r) = 0.$$

(2b) As in question (2a) we must get:

$$a = \sup_{-\frac{1+r}{u-(1+r)} \le \hat{\alpha} \le \frac{1+r}{1+r-d}} \mathbb{E}\left\{\left(\hat{\alpha}\xi + (1-\hat{\alpha})(1+r)\right)^q\right\}.$$

which is independent of x.

We are to compute the maximum. We let

$$J(\hat{\alpha}) = \mathbb{E}\left\{\left(\hat{\alpha}\xi + (1-\hat{\alpha})(1+r)\right)^q\right\}$$

$$= \underset{1/2}{p}\left\{\hat{\alpha}u + (1-\hat{\alpha})(1+r)\right\}^q + \underset{1/2}{(1-p)}\left\{\hat{\alpha}d + (1-\hat{\alpha})(1+r)\right\}^q$$

J is differentiable and

$$J'(\hat{\alpha}) = [u-(1+r)]\frac{q}{2}\left\{\hat{\alpha}(u-(1+r)) + 1+r\right\}^{q-1}$$
$$+ [d-(1+r)]\frac{q}{2}\left\{\hat{\alpha}(d-(1+r)) + 1+r\right\}^{q-1}$$

Therefore $J'(\hat{\alpha}) = 0 \iff [u-(1+r)]^{\frac{1}{q-1}}\left[\hat{\alpha}(u-(1+r)) + 1+r\right]$

$$= [1+r-d]^{\frac{1}{q-1}}\left[\hat{\alpha}(d-(1+r)) + 1+r\right].$$

This is equivalent to

$$J'(\hat{\alpha}) = 0 \iff \hat{\alpha} = \underbrace{[1+r]\frac{-[u-(1+r)]^{1/q-1} + [(1+r)-d]^{1/q-1}}{[u-(1+r)]^{q/q-1} + [1+r-d]^{q/q-1}}}_{\alpha^*}$$

Therefore:

$$\alpha^*[u-[1+r]] + 1+r = [1+r]\left[1 + \frac{(1+r-d)^{1/q-1}(u-(1+r)) - (u-(1+r))^{q/q-1}}{[u-(1+r)]^{q/q-1} + [1+r-d]^{1/q-1}}\right]$$

(1) $$= \frac{[1+r][1+r-d]^{1/q-1}\left[\underbrace{1+r-d + u - (1+r)}_{u-d}\right]}{[u-(1+r)]^{q/q-1} + [1+r-d]^{q/q-1}} \geq 0$$

Similarly

$$\alpha^*\{d-(1+r)\}+1+r=(1+r)\left\{1+\frac{(1+r-d)(u-(1+r))^{1/q-1}-(1+r-d)^{q/q-1}}{(u-(1+r))^{q/q-1}+(1+r-d)^{q/q-1}}\right\}$$

(2)

$$=\frac{(1+r)[u-(1+r)]^{1/q-1}}{(u-(1+r))^{q/q-1}+(1+r-d)^{q/q-1}}(u-d)\geq 0.$$

Therefore α^* is admissible.

Moreover, it is well-checked that J has a unique maximum as in question (2b). It must be the zero of the derivative, that is α^*.

We compute the reward associated with α^*:

by (1) + (2)

$$c_{max}=\frac{1}{2}\left[\alpha^*(u-(1+r))+1+r\right]^q+\frac{1}{2}\left[\alpha^*(d-(1+r))+1+r\right]^q$$

$$=\frac{1}{2}(1+r)^q(u-d)^q\frac{(u-(1+r))^{q/q-1}+(1+r-d)^{q/q-1}}{\left[(u-(1+r))^{q/q-1}+(1+r-d)^{q/q-1}\right]^q}$$

$$=\frac{1}{2}(1+r)^q(u-d)^q\frac{1}{\left[(u-(1+r))^{q/q-1}+(1+r-d)^{q/q-1}\right]^{q-1}}$$

(c)

$$\mathbb{E}\xi=\frac{1}{2}u+\frac{1}{2}d=1+\mu$$

$$V[\xi]=\mathbb{E}\left[(\xi-(1+\mu))^2\right]=\frac{1}{2}(u-(1+\mu))^2+\frac{1}{2}(d-(1+\mu))^2$$

$$=\frac{1}{2}a^2+\frac{1}{2}a^2=a^2.$$

$\mu>r$ would make sense to emphasize the fact that, in the mean, the risky asset should "pay" more than the non-risky asset. It would be a kind of reward for in counterpart of the risk.

(d) if $\mu=r$

$$c = \left(\frac{1+r}{2}\right)^q \frac{(2a)^q}{\left[a^{q/q-1} + a^{q/q-1}\right]^{q-1}}$$

$$= \left(\frac{1+r}{2}\right)^q \cdot \frac{(2a)^q}{2^{q-1} a^q} = (1+r)^q.$$

The investment consists in putting all the money in the non-risky asset. This is the reason why we do not see a. Indeed, since the mean benefit of the risky asset is equal to the benefit of the non-risky, the agent prefers to invest in the non-risky asset, which is secure.

(e). We let $\Delta t = T/N$. $r = \rho \Delta t$ $\mu - r = \delta \Delta t$ $a = \sigma\sqrt{\Delta t}$

then

$$c = \left(\frac{1+\rho\Delta t}{2}\right)^q \frac{2^q \sigma^q \Delta t^{q/2}}{\left[\left[\delta\Delta t + \sigma\sqrt{\Delta t}\right]^{q/q-1} + \left[\sigma\sqrt{\Delta t} - \delta\Delta t\right]^{q/q-1}\right]^{q-1}}$$

$$= \left(\frac{1+\rho\Delta t}{2}\right)^q \frac{2^q \sigma^q \Delta t^{q/2}}{\sigma^q \Delta t^{q/2} \left[\left(1 + \frac{\delta}{\sigma}\sqrt{\Delta t}\right)^{q/q-1} + \left(1 - \frac{\delta}{\sigma}\sqrt{\Delta t}\right)^{q/q-1}\right]^{q-1}}$$

Now:

$$\left(1+\frac{d}{\sigma}\sqrt{\Delta t}\right)^{q/q-1} = 1 + \frac{d}{\sigma}\frac{q}{q-1}\sqrt{\Delta t} + \frac{1}{2}\frac{d^2}{\sigma^2}\Delta t \underbrace{\frac{q}{q-1}\left(\frac{q}{q-1}-1\right)}_{\frac{q}{(q-1)^2}} + O\left(\Delta t^{3/2}\right)$$

$$\left(1-\frac{d}{\sigma}\sqrt{\Delta t}\right)^{q/q-1} = 1 - \frac{d}{\sigma}\frac{q}{q-1}\sqrt{\Delta t} + \frac{1}{2}\frac{d^2}{\sigma^2}\Delta t \frac{q}{(q-1)^2} + O\left(\Delta t^{3/2}\right)$$

Therefore:

$$\left(1+\frac{d}{\sigma}\sqrt{\Delta t}\right)^{q/q-1} + \left(1-\frac{d}{\sigma}\sqrt{\Delta t}\right)^{q/q-1}$$

$$= 2 + \frac{d^2}{\sigma^2}\Delta t \frac{q}{(q-1)^2} + O\left(\Delta t^{3/2}\right)$$

So that

$$\left[\left(1+\frac{d}{\sigma}\sqrt{\Delta t}\right)^{q/q-1} + \left(1-\frac{d}{\sigma}\sqrt{\Delta t}\right)^{q/q-1}\right]^{q-1}$$

$$= 2^{q-1}\left[1 + \frac{d^2}{2\sigma^2}\frac{q}{(q-1)^2}\Delta t + O\left(\Delta t^{3/2}\right)\right]^{q-1}$$

$$= 2^{q-1}\left[1+\frac{d^2}{2\sigma^2}\frac{q}{q-1}\Delta t + o(\Delta t^{3/2})\right]$$

Similarly

$$(1+\rho\Delta t)^q = 1+q\rho\Delta t + o(\Delta t^2).$$

Therefore

$$c = \frac{1+q\rho\Delta t + o(\Delta t^2)}{1+\frac{d^2}{2\sigma^2}\frac{q}{q-1}\Delta t + o(\Delta t^{3/2})}$$

$$= 1+\left(q\rho - \frac{d^2}{2\sigma^2}\frac{q}{q-1}\right)\Delta t + o(\Delta t^{3/2})$$

Now:

$$c^N = \exp\left\{\frac{T}{\Delta t}\ln\left\{1+\left(q\rho-\frac{d^2}{2\sigma^2}\frac{q}{q-1}\right)\Delta t + o(\Delta t^{3/2})\right\}\right\}$$

$$T\left(q\rho - \frac{d^2}{2\sigma^2}\frac{q}{q-1}\right) + o(\Delta t^{1/2})$$

$$\underset{\substack{N\to+\infty \\ (\Leftrightarrow \Delta t\to 0)}}{\sim} \exp\left\{T\left(q\rho - \frac{d^2}{2\sigma^2}\frac{q}{q-1}\right)\right\}.$$

When $N\to\infty$ the optimal reward is

$$x^q \exp\left\{T\left(q\rho - \frac{d^2}{2\sigma^2}\frac{q}{q-1}\right)\right\}.$$