Université de Nice - M2 - Mathmods - Stochastic Control

EXAM

FEBRUARY 20, 2015 – DURATION : 2H30 – NO DOCUMENTS.

We are given a financial market made of two assets, a risky one and a non-risky one. The market evolves in discrete time over N periods. The spot price of the non-risky asset at times $0, 1, \ldots, N$ is denoted by $S_0^0, S_1^0, \ldots, S_N^0$. Similarly, the spot price of the risky asset at times $0, 1, \ldots, N$ is denoted by $S_0, S_1, \ldots, S_N$.

The dynamics of the non-risky asset are given by :

$$\forall n \in \{0, \ldots, N\}, \quad S_n^0 = S_0(1+r)^n,$$

where $r > 0$, whereas the dynamics of the risky asset follow a binomial model :

$$\forall n \in \{0, \ldots, N\}, \quad S_n = S_0 \xi_1 \ldots \xi_n,$$

where $\xi_1, \ldots, \xi_N$ are N independent and identically distributed random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with distribution :

$$\forall n \in \{0, \ldots, N\}, \quad \mathbb{P}(\xi_n = u) = 1 - \mathbb{P}(\xi_n = d) = p,$$

for $p \in (0,1)$ and $d < 1 + r < u$.

We assume that, for any $n = 0, \ldots, N-1$, the detention of k shares of the risky asset between times n and $n+1$ pays, at time $n+1$, a dividend equal to kD, where $D \geq 0$ is the dividend per share.

The filtration generated by the sequence $(\xi_n)_{n=0,\ldots,N}$ is denoted by $(\mathcal{F}_n)_{n=0,\ldots,N}$.

(1) Assume that, at some time $n = 0, \ldots, N-1$, a financial agent has a capital $W_n > 0$ and invests a proportion $\hat{\alpha}_n \in [0,1]$ of the capital in the risky asset and, then, the proportion $(1 - \hat{\alpha}_n)$ in the non-risky asset. Show that the wealth at the next time $n+1$ writes

$$W_{n+1}^{\hat{\alpha}_n} = \hat{\alpha}_n W_n \big(\xi_{n+1} + \frac{D}{S_n}\big) + (1 - \hat{\alpha}_n) W_n (1+r),$$

where the superscript $\hat{\alpha}_n$ in the left-hand side indicates that the wealth at time $n+1$ depends on the strategy of investment chosen by the agent at time n. Check that $W_{n+1}^{\hat{\alpha}_n}$ is non-negative.

(2) Assume for the moment that $N = 1$ and that the financial agent aims at maximizing $\mathbb{E}[U(W_1^{\hat{\alpha}_0})]$ over $\hat{\alpha}_0$, for some utility function $U : (0, +\infty) \to \mathbb{R}$, for an initial capital $W_0 = x > 0$ and for an initial spot price $S_0 = s > 0$.

(a) Taking $U(x) = x^q$ for some $q \in (0,1)$, show that there exists a ~~function~~ non-increasing function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sup_{\hat{\alpha} \in [0,1]} \mathbb{E}[U(W_1^{\hat{\alpha}})] = c(s) x^q.$$

(b) Prove that there exists a unique $\alpha^*(s) \in [0,1]$ such that

$$\alpha^*(s) = \mathrm{argmax}_{\hat{\alpha}\in[0,1]} \mathbb{E}[U(W_1^{\hat{\alpha}})],$$

and check that it does not depend on x.

(3) We now return to the case when $N \geq 1$. The goal is to maximize $\mathbb{E}[U(W_N^{\hat{\alpha}})]$ over the strategies $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_0, \ldots, \hat{\alpha}_{N-1})$ that are adapted to the filtration $(\mathcal{F}_n)_{n=0,\ldots,N-1}$.

To this end, we let the value function U_n at time n be a function of both the wealth x of the agent at time n and the spot price of the risky asset s at time n :

$$U_n(x,s) = \sup_{(\hat{\alpha}_n,\ldots,\hat{\alpha}_{N-1})} \mathbb{E}\big[U\big(W_N^{(\hat{\alpha}_n,\ldots,\hat{\alpha}_{N-1})}\big)\big],$$

with the prescription that $W_n = x$ and $S_n = s$.

(a) What are the admissible values for s at time n (in terms of S_0, d, u and n)?

(b) Prove that U_N does not depend on s.

(c) Propose (without any proof) a suitable version of the dynamic programming principle.

(4) In this question, take for granted the dynamic programming principle proposed right above.

(a) Prove that, for any $n \in \{0,\ldots,N\}$, there exists a non-increasing function $c_n : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$U_n(x,s) = c_n(s)x^q.$$

(b) Express $c_n(s)$ in terms of $c_{n+1}(s)$.

(c) Prove that, at any time n, there is one and only optimal choice for the financial agent to allocate his/her capital. Describe the information that is needed to proceed with such an allocation.

(5) In this question, we suppose $D = 0$ and choose $p = 1/2$. We also allow $\hat{\alpha}$ to be in $[-(1+r)/(u-(1+r)), (1+r)/(1+r-d)]$.

(a) Show that, with the admissible values for the allocation, the wealth remains non-negative.

(b) Show that Question (2a) holds true, but with c independent of s given by

$$c = \frac{(1+r)^q}{2} \frac{(u-d)^q}{\left[(u-(1+r))^{q/(q-1)} + ((1+r)-d)^{q/(q-1)}\right]^{q-1}}.$$

(c) Choose $u = 1+\mu+a$ and $d = 1+\mu-a$. Check that $1+\mu$ is the mean of ξ and a^2 is the variance of ξ. Why would it make sense to assume $\mu > r$?

(d) Show that $c = (1+r)^q$ when $\mu = r$. Explain why it does not depend on a.

(e) Let $\Delta t = T/N$ be the time step of a discretization of an interval $[0,T]$. choose $r = \rho\Delta t$, $\mu - r = \lambda\Delta t$ and $a = \sigma\sqrt{\Delta t}$. Show that

$$c^N \sim_{N\to\infty} \exp\Big[\Big(q\rho - \frac{q}{2(q-1)}\frac{\lambda^2}{\sigma^2}\Big)T\Big].$$

What can you say about the optimal ~~wealth~~ when N tends to ∞?

 reward.

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EXAM

MARCH 19, 2015 – DURATION: 3H00 – NO DOCUMENTS.

The exam consists of two parts, which are almost independent.

PART A

On a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, equipped with a one-dimensional Brownian motion $(W_t)_{t\geq 0}$ with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, consider a controlled process of the form:

$$(\square) \qquad dX_t = \alpha_t dt + dW_t, \quad t \in [0,T],$$

the initial value $X_0 = x_0$ being prescribed and the control process $(\alpha_t)_{t\in[0,T]}$ being $(\mathcal{F}_t)_{t\in[0,T]}$-progressively measurable and satisfying

$$\mathbb{E}\int_0^T |\alpha_t|^2 dt < \infty.$$

Given a smooth bounded function $g : \mathbb{R} \to \mathbb{R}$, with bounded derivatives of any order, we then let $J((\alpha_t)_{t\in[0,T]})$ be the cost functional:

$$(\square\square) \qquad J\big((\alpha_t)_{t\in[0,T]}\big) = \mathbb{E}\Big[g(X_T) + \frac{1}{2}\int_0^T |\alpha_t|^2 dt\Big].$$

The goal is to identify the optimal path(s) minimizing J.

(1) From the general form given in the course, show that the Hamilon-Jacobi-Bellman equation here writes:

$$(\star) \qquad \begin{aligned} &\partial_t u(t,x) + \frac{1}{2}\partial^2_{xx} u(t,x) - \frac{1}{2}|\partial_x u(t,x)|^2 = 0, \quad (t,x) \in [0,T]\times\mathbb{R}, \\ &u(T,x) = g(x), \quad x \in \mathbb{R}. \end{aligned}$$

(2) Show that $u \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ solves $(\star)$ if and only if $v = \exp(-u)$ solves

$$(\star\star) \qquad \begin{aligned} &\partial_t v(t,x) + \frac{1}{2}\partial^2_{xx} v(t,x) = 0, \quad (t,x) \in [0,T], \times\mathbb{R} \\ &v(T,x) = \exp(-g(x)), \quad x \in \mathbb{R}. \end{aligned}$$

(3) Consider the function

$$w(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\big[-g(x+\sqrt{T-t}\,y)\big]\underbrace{\exp\big(-\frac{y^2}{2}\big)}_{f_Y}dy, \quad (t,x)\in[0,T]\times\mathbb{R}.$$

(a) Show that w is continuous in (t,x) and has derivatives of any order in x that are continuous in (t,x). Check that all the derivatives (in x) are bounded.

(b) Prove that w is differentiable in the parameter t on $[0,T)$, $\partial_t w$ being continuous on $[0,T)\times\mathbb{R}$.

$E[\cancel{g}(x+\sqrt{T-t}\cdot Y)]$ $Y \sim \mathcal{N}(0,1)$

$\boxed{\dfrac{\partial B_t}{\partial t}}$

$\|$

$E[-g(x+B_T-B_t))]$

$\mathbb{E}[\Phi(X)] =$

$X = x + \sqrt{T-t}\, Y$

(4) Show that w may be written under the form

$$w(t,x) = \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} \exp\big[-g(y)\big] \exp\Big(-\frac{(x-y)^2}{2(T-t)}\Big) dy,$$

and solves $(\star\star)$ on $[0,T) \times \mathbb{R}$.

By a continuity argument, we could prove that w is differentiable in time at $t = T$ as well and that $w \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$. We admit this claim. Below, we let:

$$z(t,x) = -\ln\big(w(t,x)\big), \quad (t,x) \in [0,T] \times \mathbb{R}.$$

(5) Consider now $(\square)$ for some control process $(\alpha_t)_{t\in[0,T]}$. Using Itô's formula, prove that

$$J\big((\alpha_t)_{t\in[0,T]}\big) = z(0,x_0) + \frac{1}{2}\mathbb{E}\int_0^T |\alpha_t + \partial_x z(t,X_t)|^2 dt.$$

(6) Deduce that there exists a unique path $(X_t^\star)_{t\in[0,T]}$ minimizing J (with the given initial condition). Characterize it as the solution of a uniquely solvable SDE (prove that the SDE admits a unique solution but don't try to find the explicit form of the solution).

(7) Prove that $(\partial_x z(t,X_t^\star))_{t\in[0,T)}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T)}$. Deduce that there exists a constant c, possibly depending upon x_0, such that, for all $t \in [0,T]$,

$$(\star\star\star) \qquad \mathbb{E}[X_t^\star] = x_0 + tc.$$

Part B

We now consider $(\square\square)$ but with $g(x) = \frac{1}{2}x^2$.

(1) Find a solution z' to $(\star)$ such that $\partial_x z'(t,\cdot)$ is a linear function in x for any t.

(2) Consider now $(\square)$. Using Itô's formula, prove that

$$J\big((\alpha_t)_{t\in[0,T]}\big) = z'(0,x_0) + \frac{1}{2}\mathbb{E}\int_0^T |\alpha_t + \partial_x z'(t,X_t)|^2 dt.$$

(3) Deduce that there exists a unique path $(X_t^\star)_{t\in[0,T]}$ minimizing J. Find the explicit shape of $(X_t^\star)_{t\in[0,T]}$.

(4) Prove directly that $(\star\star\star)$ holds true and find the value of the constant c therein.

(5) Replace now the condition $g(x) = \frac{1}{2}x^2$ by $g(x) = -\frac{1}{2}x^2$ and take $T = 1 - \varepsilon$ for some $\varepsilon \in (0,1)$.

(a) Compute the new value of c in $(\star\star\star)$.

(b) What happens when $\varepsilon \to 0$?

(c) In order to explain the above phenomenon, choose $x_0 = 0$, $T = 1$ and $\alpha_t = A$, for all $t \in [0,1]$, in $(\square)$, for some large real $A > 0$. Show that the cost to $(\alpha_t)_{t\in[0,1]}$ is constant with A.

(d) Deduce that the cost functional may remain bounded along sequence of controls $(\alpha_t)_{t\in[0,T]}$ such that the sequence of kinetic energies $\mathbb{E}\int_0^1 |\alpha_t|^2 dt$ is unbounded.