Université de Nice Sophia-Antipolis
MathMods

2012-2013

Exam - Advanced Numerics for Computational Finance
19th February 2013 – 3 hours

Exercise 1- Euler scheme

For any $n \in \mathbb{N}$, we denote by $(X_k^{(n)})_{1 \leq k \leq n}$ a sequence of independent gaussian random variables with mean 0 and variance $\frac{1}{n}$. We set $S_0^{(n)} = 0$ and

$$S_k^{(n)} = \sum_{i=1}^{k} X_i^{(n)} \quad k \geq 1$$

$$Y_1^{(n)} = \sum_{k=0}^{n-1} S_k^{(n)} X_{k+1}^{(n)}$$

$$Z_1^{(n)} = \sum_{k=0}^{n-1} \left(S_k^{(n)} + \frac{X_{k+1}^{(n)}}{2} \right) X_{k+1}^{(n)}$$

a) Does the random variable $S_n^{(n)}$ converge as $n \to \infty$? In which sense? What is the limit?

b) Does the random variable $Y_1^{(n)}$ converge as $n \to \infty$? In which sense? What is the limit?

c) Does the random variable $Z_1^{(n)}$ converge as $n \to \infty$? In which sense? What is the limit?

Exercise 2

Let $(W_t, t \geq 0)$ be a one-dimensional Brownian motion and $T_0 < T_1 < T_2$ three fixed deterministic times. We set:

$$Y_{\alpha,\beta} = W_{T_1} - \alpha W_{T_0} - \beta W_{T_2}.$$

1) What is the law of $Y_{\alpha,\beta}$?
2) Which conditions on α and β imply that

a) $Y_{\alpha,\beta}$ and W_{T_0} are independent

b) $Y_{\alpha,\beta}$ and W_{T_2} are independent

c) $Y_{\alpha,\beta}$ is independent of W_{T_0} and W_{T_2}

3) Write W_{T_1} as a function of W_{T_0}, W_{T_2} and $Y_{\alpha,\beta}$.
Assume that you have already simulated a Brownian Motion at time T_0, $T_2, \cdots, T_{2N}$ and you want to know the path on a finer grid, that is $T_0 < T_1 < T_2 < \cdots < T_{2N-1} < T_{2N}$.
Knowing $W_{T_0}, W_{T_2}, \cdots, \cancel{B}_{T_{2N}}$ W, give precisely the algorithm to generate $W_{T_1}, W_{T_3}, \cdots, W_{T_{2N-1}}$.

Problem- First Hitting Times

Part A- Classical functions associated to diffusion

Let $(X_t, 0 \leq t \leq T)$ be the solution of the homogeneous (i.e the coefficients do not depend on time) stochastic differential equation:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t. \tag{1}$$

Let τ_a be the first hitting time of a by the diffusion process X

$$\tau_a^X = \inf\{t \geq 0, X_t = a\}.$$

We denote $\tau_{a,b}^X = \inf\{\tau_a^X, \tau_b^X\}$.

a) Find the infinitesimal generator L associated to this diffusion process. We recall the definition of the infinitesimal generator $Lf(x) = \lim_{\varepsilon \to 0} \mathbb{E}^{0,x}\left[\frac{f(X_\varepsilon)-f(x)}{\varepsilon}\right]$

b) Characterize the functions s such that $s(X_t)$ is a local martingale. (such functions are known as scale functions)

c) If the initial condition x is not between a and b, simplify $\tau_{a,b}^X$.

d) Otherwise, (that is $x \in (a,b)$ or $x \in (b,a)$), assuming the optimal stopping theorem is valid, find a relation between $\mathbb{P}^{0,x}\left(\tau_a^X < \tau_b^X\right)$ and the scale function s.

Part B- Brownian motion with constant drift

e) Apply the previous result to a Brownian motion with constant drift to find

$$\mathbb{P}^{0,x}\left[\sup_{t\geq 0}(W_t + \delta t) > b\right]$$

where δ is a fixed real number (discuss the result in function of δ).

f) Applying the "reflection" principle, prove that:

$$\mathbb{P}\left(\sup_{s\in[0,t]}(W_s) \geq a\right) = \mathbb{P}\left(|G| \geq \frac{a}{\sqrt{t}}\right),$$

where G is a standard gaussian random variable $\mathcal{N}(0,1)$.

g) Give the probability density function of the first hitting time of a constant by a standard one dimensional Brownian motion.

h) Applying Girsanov's Theorem, give the probability density function of the first hitting time of a constant by a Brownian motion with constant drift, that is $W_t + \delta t$.

Université de Nice Sophia-Antipolis 2013-2014
MathMods

Exam - Advanced Numerics for Computational Finance
January 16th, 2014 – 3 hours

J. Inglis - E. Tanré

Throughout the exam, we will work on an underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

1 Part A: Short questions

1. Let n and d be positive integers. Consider the SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \qquad t \in [0, T], \quad X_0 = \xi, \tag{1}$$

where $(W_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued Brownian motion starting from 0 (adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$), ξ is a random variable, $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, and $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$.

   (a) State the precisely the classical conditions under which the SDE (1) has a unique (strong) solution $X_t \in \mathbb{R}^n$ for all $t \geq 0$.

   (b) Describe the Monte-Carlo method to approximate $\mathbb{E}[f(X_T)]$ for some function $f : \mathbb{R}^n \to \mathbb{R}$, supposing that we use an Euler scheme to sample approximately from the law of X_T. Given that the Monte-Carlo number is M, and that the time step in the Euler scheme is $\delta > 0$, how many times must we sample a normal random variable to complete the estimation (supposing $n = d = 1$)?

   (c) Decompose the error in the Monte-Carlo method described in (b) into the discretization error and the statistical error. Give the names of two theorems (without precise statements) that can be applied to control the two error terms.

2. Let $(W_t^x)_{t\geq 0}$ be a one-dimensional Brownian motion such that $W_0^x = x \in \mathbb{R}$ almost surely and

$$M_t^x = \sup_{0\leq s\leq t} W_s^x.$$

   (a) State and prove the reflection principle for Brownian motion.

   (b) Use the reflection principle to derive the density of M_t^x.

   (c) Give an explicit scheme to simulate the law of M_t^x.

$$f_{M_t^x}(a) = \frac{d}{da}\mathbb{P}(M_t^x \leq a) = \frac{d}{da}\left\{1 - \mathbb{P}(M_t^x \geq a)\right\} = -\frac{d}{da}\mathbb{P}(M_t^x \geq a)$$

$$= -\frac{d}{da} 2\mathbb{P}(B_t^x \geq a) = -\frac{d}{da}\underbrace{\frac{2}{\sqrt{2\pi}}\int_0^x \frac{1}{\sqrt{t}}\exp\left(-\frac{s}{2t}\right)ds}$$

$$= \frac{-2}{\sqrt{2\pi t}}\exp\left(-\frac{x}{2t}\right)$$

$P(M_t \geq a, W \leq a-b)$

$= P(M_t \geq a) - P($

(d) Prove that

$$\mathbb{P}\left(M_T^x \geq a, W_T^x \leq a-b\right) = \mathbb{P}\left(W_T^x \geq a+b\right),$$

for all $x, a, b \in \mathbb{R}$ and $T > 0$. Use this to deduce that

$$\mathbb{P}\left(M_T^x \geq z | W_T^x = y\right) = \exp\left(-\frac{2}{T}(z-x)(z-y)\right),$$

for all $x, y, z \in \mathbb{R}$ such that $x < z$, $y < z$ and all $T > 0$.

3. Suppose that we are in the Black-Scholes set up: we have a risky asset with price process $(S_t)_{t\geq 0}$ that satisfies

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad t \geq 0,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and $(W_t)_{t\geq 0}$ is a standard 1-dimensional Brownian motion, as well as a risk-free asset with price process $(B_t)_{t\geq 0}$ evolving according to

$$dB_t = rB_t dt, \quad t \geq 0,$$

where $r > 0$.

Let $V(t, x)$, $t \geq 0$, $x \in \mathbb{R}$ be the price of an option on the asset at time t and when $S_t = x$. Suppose the option has maturity T and payoff function given by $\Psi : \mathbb{R} \to \mathbb{R}$. State the general pricing formula for V.

Determine an explicit formula for the value of the option when

$$\Psi(x) = \begin{cases} 1 & \text{if } x \geq K \\ 0 & \text{otherwise} \end{cases},$$

where $K > 0$ is a constant.

2 Part B: Long question

Consider the stochastic differential equation in one dimension given by,

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \qquad t \in [0, 1], \tag{2}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion starting from 0, and μ and σ are functions $: \mathbb{R} \to \mathbb{R}$. Suppose that the initial condition is given by

$$X_0 = \xi,$$

where ξ is an almost surely finite random variable on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent of $(W_t)_{t\geq 0}$. Suppose that (2) has a unique solution for all $t \in [0, 1]$, and that we would like to approximate this solution.

6. Prove by induction on k, that for any $N \in \mathbb{N}$ and $\omega \in \Omega_N$, it holds that

$$|\bar{X}^\delta_{k\delta}(\omega)| \geq r_N^{\alpha^{k-1}},$$

for all $k \in \{1, \ldots, N\}$. In particular, use this to deduce that

$$|\bar{X}^\delta_1(\omega)| \geq 2^{\alpha^{N-1}} \tag{4}$$

$\geq r_N^{\alpha^{N-1}} \geq 2^{\alpha^{N-1}}$ $1 \leq r_N$

for all $N \in \mathbb{N}$, $\omega \in \Omega_N$.
Hint: A good place to start is to note that under the induction hypothesis, it holds that

$$|\bar{X}^\delta_{k\delta}| \geq r_N^{\alpha^{k-1}} \geq r_N \geq C \geq 1,$$

since $\alpha > 1$ and by definition of r_N. You may also like to use the elementary inequality $|a + b| \geq \max\{|a|, |b|\} - \min\{|a|, |b|\}$ for all $a, b \in \mathbb{R}$.

7. Use the Brownian scaling property to prove that, by the definition of Ω_N,

$$\mathbb{P}(\Omega_N) \geq \frac{\theta e^{-2}}{4} 2^{-N} N^{-N} \exp\left(-NK^2(r_N + K)^2\right),$$

for all $N \in \mathbb{N}$ (where θ is given by (3)). You may also use without proof the inequalities

$$\mathbb{P}(|Z| \geq x) \geq \frac{xe^{-x^2}}{4}, \qquad \mathbb{P}(|Z| \in [x, 2x]) \geq \frac{xe^{-2x^2}}{2},$$

for any $x \geq 0$, where $Z \sim \mathcal{N}(0, 1)$ is a standard normally distributed random variable.

8. Deduce that there exist constants $c_1, c_2 > 0$ and $\gamma > 1$ (independent of N) such that whenever N is large enough, it holds that

$$\mathbb{P}(\Omega_N) \geq c_1 \exp(-c_2 N^\gamma). \tag{5}$$

9. Use (4) and (5) to prove that

$$\lim_{N\to\infty} \mathbb{E}[|\bar{X}^\delta_1|] = \infty,$$

and thus that

$$\lim_{N\to\infty} \mathbb{E}[|\bar{X}^\delta_1|^p] = \infty, \tag{6}$$

for any $p \geq 1$.

9. Finally, returning to equation (2), suppose that the true solution is such that

$$\mathbb{E}[|X_1|^p] < \infty$$

for some $p \in [1, \infty)$. Explain why (6) shows that the Euler scheme fails to converge in both the strong and the weak sense under the given conditions on μ and σ. Use the above results to give a simple example of an SDE with continuous coefficients such that $\mathbb{E}[|X_1|] < \infty$, but for which the Euler scheme does not converge.

Université de Nice-Sophia Antipolis
MathMods

2014-2015
January 20th, 2015 – 3 hours

Exam - Numerical Methods in Probability for Mathematical Finance

J. Inglis - E. Tanré

Course notes forbidden

1 Part A: Shorter questions

1. Recall that the Laplace transform of a non-negative real-valued random variable X is given by
$$\varphi_X(\lambda) := \mathbb{E}\left(e^{-\lambda X}\right),$$
for $\lambda \geq 0$.

   (a) Compute the Laplace transform of the random variable X distributed according to an exponential distribution with parameter $\theta > 0$ (so that $\mathbb{E}(X) = \theta^{-1}$).

   (b) Let $(X_k)_{k\in\{1,\dots,N\}}$ be a family of independent random variables with the same exponential probability distribution with parameter $\theta > 0$. Set
$$S_N := \sum_{k=1}^{N} X_k.$$
   Compute the Laplace transform of S_N. Use this to confirm that the probability density function of S_N is
$$p_N(x) := \frac{\theta^N}{(N-1)!}x^{N-1}e^{-\theta x}, \quad x \geq 0.$$

   (c) Let M be the smallest integer N such that $S_{N+1} > \theta$. Show that this random variable has a Poisson distribution with parameter θ.

   (d) Propose a simulation method of the Poisson distribution with parameter θ which requires samples only of the uniform distribution on $[0, 1]$.

2. Let $n \geq 1$ and $d \geq 1$ be two fixed integers. Consider the SDE
$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \quad t \geq 0, \tag{1}$$
with initial condition $X_0 \in \mathbb{R}^n$, where $a : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $b : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ and $(W_t)_{t\geq 0}$ is a standard $\mathbb{R}^d$-valued Brownian motion.

(a) Let $T > 0$ be fixed. Describe how to construct the Euler approximation of the solution to the SDE (1) with time step discretization $\delta > 0$ on the time interval $[0, T]$. Denote by $(\bar{X}^{\delta}_{l\delta})_{l \in \{0,1,\dots m\}}$ this approximation.

(b) State carefully the Strong Convergence Theorem for the Euler Scheme, making precise the assumptions on the coefficients a and b.

(c) How many Gaussian variables are necessary to obtain M samples of the law of $\bar{X}^{\delta}_T$?

3. Suppose that we are in the Black-Scholes set up: we have a risky asset with price process $(S_t)_{t\geq 0}$ that satisfies

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad t \geq 0,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and $(W_t)_{t\geq 0}$ is a standard 1-dimensional Brownian motion, as well as a risk-free asset evolving according to the risk free rate $r > 0$. State the general pricing formula for the price $V(t, x)$ of an option (at time t and when the price of the underlying is x) that pays $\Psi(S_T)$ at time T.

An Asian option is an option that has a more general payoff function depending on the whole path $(S_t)_{t\in[0,T]}$ (not only on the price S_T at time T) given by

$$\Psi\left((S_t)_{t\in[0,T]}\right) = \max\left\{\frac{1}{T}\int_0^T S_t dt - K, 0\right\},$$

for some $K > 0$ i.e. the holder of the option receives a payoff at T if the average price of the underlying asset on $[0, T]$ is bigger than K. Using the general pricing formula with this payoff function, suggest a Monte-Carlo scheme to approximate the price of this option when $S_0 = x$.

4. (a) Fix $a, \lambda \in \mathbb{R}$, and suppose that $u \in \mathcal{C}^2(\mathbb{R})$ is a solution to the ODE

$$\begin{cases} \mathcal{L}u(x) &= \lambda u(x), \qquad x < a, \\ u(a) &= 1, \end{cases} \tag{2}$$

where the infinitesimal generator $\mathcal{L}$ is given by

$$\mathcal{L}u(x) := \frac{\sigma^2}{2}u''(x) + b(x)u'(x),$$

with $b : \mathbb{R} \to \mathbb{R}$ and $\sigma > 0$ a constant. By applying Itô's formula to $e^{-\lambda t}u(X_t)$ where

$$dX_t = b(X_t)dt + \sigma dW_t, \tag{3}$$

show that u must be given by

$$u(x) = \mathbb{E}\left(e^{-\lambda\tau_a}|X_0 = x\right), \quad \forall x < a,$$

where $\tau_a := \inf\{t > 0 : X_t \geq a\}$. Clearly state any theorems that you use, and explain why they are applicable.

(b) Describe two algorithms to approximate $\tau_a = \inf\{t > 0 : X_t \geq a\}$, with $(X_t)_{t\geq 0}$ given by (3) and $X_0 = x < a$, and state what assumptions on the drift b are sufficient for these approximations to converge. Give the theoretical rates of convergence of the two schemes.

2 Part B: Long question

Consider an $\mathbb{R}^N$-valued random variable X with the following property: there exists a constant $C > 0$ such that for any $f \in \mathcal{C}^1(\mathbb{R}^N)$ it holds that

$$\mathbb{E}\left(f^2(X)\log\left(\frac{f^2(X)}{\mathbb{E}(f^2(X))}\right)\right) \leq C\mathbb{E}\Big(|\nabla f|^2(X)\Big). \tag{4}$$

Here, as usual, $\nabla f(x) := (\partial_{x_1} f(x), \ldots, \partial_{x_N} f(x)) \in \mathbb{R}^N$ for $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and $|\cdot|$ denotes the usual Euclidian distance in $\mathbb{R}^N$ i.e.

$$|x|^2 = \sum_{i=1}^N x_i^2, \qquad \forall x = (x_1, \ldots, x_N) \in \mathbb{R}^N.$$

When (4) holds, we say that **the law of X satisfies a logarithmic Sobolev inequality with constant C.**

1. Use (4) to show that for all $F \in \mathcal{C}^1(\mathbb{R}^N)$ such that $|\nabla F|(x) \leq 1$ for all $x \in \mathbb{R}^N$, it holds that
$$\mathbb{E}\left(e^{\lambda(F(X)-\mathbb{E}[F(X)])}\right) \leq e^{\frac{C}{4}\lambda^2} \tag{5}$$
for all $\lambda \in \mathbb{R}$.

   Hint: Apply the inequality (4) to $f = e^{\frac{1}{2}\lambda F}$, and note that if $H(\lambda) := \frac{1}{\lambda}\log\mathbb{E}(e^{\lambda F(X)})$, then
$$\lambda^2 H'(\lambda)\mathbb{E}\left(e^{\lambda F(X)}\right) = \lambda\mathbb{E}\left(F(X)e^{\lambda F(X)}\right) - \mathbb{E}\left(e^{\lambda F(X)}\right)\log\mathbb{E}\left(e^{\lambda F(X)}\right).$$

   This will yield a differential inequality for H, which can be solved by integrating between 0 and λ.

2. Use (5) to show that for all $r > 0$,

$$\mathbb{P}\Big(|F(X) - \mathbb{E}(F(X))| \geq r\Big) \leq 2e^{-\frac{r^2}{C}} \tag{6}$$

again for any $F \in \mathcal{C}^1(\mathbb{R}^N)$ such that $|\nabla F| \leq 1$.

Hint: Note that for any $\lambda > 0$

$$\mathbb{P}\Big(F(X) - \mathbb{E}(F(X)) \geq r\Big) = \mathbb{P}\Big(e^{\lambda(F(X) - \mathbb{E}[F(X)])} \geq e^{\lambda r}\Big),$$

to which one can apply Markov's inequality. One can note that the same argument can also be applied to $-F$.

3. Now let $(X_i)_{i\in\{1,\dots,N\}}$ be a family of independent identically distributed $\mathbb{R}$-valued random variables. Suppose that the common law of X_i satisfies a logarithmic Sobolev inequality with constant C. Use (6) to show that for any $r > 0$ and $g \in \mathcal{C}^1(\mathbb{R})$ such that $|g'| \leq 1$,

$$\mathbb{P}\left(\left|\frac{1}{N}\sum_{i=1}^{N} g(X_i) - \mathbb{E}[g(X_1)]\right| \geq r\right) \leq 2\exp\left(-\frac{Nr^2}{C}\right). \tag{7}$$

Hint: You may use without proof the fact that the law of $X = (X_1, \dots, X_N) \in \mathbb{R}^N$ also satisfies a logarithmic Sobolev inequality (4), with the same constant C.

4. Suppose that X_1 is an $\mathbb{R}$-valued random variable, and that we want to approximate $\mu = \mathbb{E}(X_1)$ by a Monte-Carlo simulation. Let $\hat{\mu}_N$ denote the Monte-Carlo approximation of μ with N simulations.

   Suppose moreover we know that the law of X_1 satisfies a logarithmic Sobolev inequality, with constant C. Use (7) to estimate the number of simulations N that we need in order to achieve an accuracy in our Monte-Carlo simulation of 0.1 with 95% confidence. In other words, give a value of N such that the probability that

$$\mu_N \in [\mu - 0.1, \mu + 0.1]$$

   is greater than 0.95. What advantage does this method of obtaining confidence intervals have over using the central limit theorem?