Exponential Families

$$\alpha(\eta)\, e^{B(\eta)\, T(x)}\, \underbrace{h(x)}_{\geq 0}$$

η parameter of family

$B(\eta) = \eta$ canonical

Exercise Binomial

$$\eta^{x}(1-\eta)^{1-x} = \exp\left[\log \eta^{x}(1-\eta)^{1-x}\right]$$

$$= \exp\left[\log \eta^{x} + \log (1-\eta)^{1-x}\right]$$

$$= \exp\left[\log \eta^{x}\right] \cdot \exp\left[\log (1-\eta)^{1-x}\right]$$

$$= \cancel{e^{x} / e^{\log \eta}\ e^{(1-\eta)}}$$

$$= \exp\left[x \log \eta\right] \exp\left[(1-x) \log(1-\eta)\right]$$

$$= \exp\left[x \log \eta + \log(1-\eta) - x \log(1-\eta)\right]$$

$$= \exp\left[x\left(\log \eta - \log(1-\eta)\right) + \log(1-\eta)\right]$$

$$= (1-\eta)\, e^{x}\, e^{\log \frac{\eta}{1-\eta}}$$

$$= \alpha(\eta)\, e^{T(x)\, B(\eta)}$$

Exercise ~~B~~ Poisson.

$$\frac{\eta^x}{x!} e^{-\eta} = e^{-\eta} e^{x \ln \eta} \frac{1}{x!}$$

$$= \alpha(\eta) e^{B(\eta) T(x)} h(x).$$

Exercise Gaussian

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} (x-\eta)^2 \right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2} - \frac{\eta^2}{2\sigma^2} + \frac{2x\eta}{2\sigma^2} \right)$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\eta^2/2\sigma^2} \right) e^{\frac{x\eta}{\sigma^2}} e^{-x^2/2\sigma^2}$$

$$\alpha(\eta) \qquad e^{B(\eta) T(x)} \qquad h(x).$$

Exercise Exponential

$$\eta \, e^{-\eta x}$$

$$\alpha(\eta) \, e^{\beta(\eta) T(x)} \, h(x)$$

Exercise Geometric

$$\eta (1-\eta)^{x-1} = \exp\left[\log \eta (1-\eta)^{x-1} \right]$$

$$= \exp\left[\log \eta + (x-1) \log(1-\eta) \right]$$

$$= \cancel{\eta} \exp\left[\log \eta + x \log(1-\eta) - \log(1-\eta) \right]$$

$$= \exp\left[\log \eta/1-\eta + x \log(1-\eta) \right]$$

$$= \frac{\eta}{1-\eta} e^{x \log(1-\eta)}$$

$$= \alpha(\eta) \, e^{\beta(\eta) T(x)} \, h(x)$$

Exercise Posterior Bernoulli

$$X \sim B(p)$$
$$p \sim \text{Beta}(\alpha, \beta)$$

$$f(\theta \mid x^n) = \frac{\mathcal{L}(x^n \mid \theta)\, \pi(\theta)}{m(x)}$$

$$\pi(p) \sim \text{Beta}(\alpha, \beta) \propto p^{\alpha-1}(1-p)^{\beta-1}$$

$$L_n(x; p) = \prod_i^n p^{x}(1-p)^{1-x} = p^{S_n}(1-p)^{n-S_n}$$

$$f(p \mid x^n) \propto p^{(S_n+\alpha)-1}(1-p)^{(n-S_n+\beta)-1}$$

$$\sim \text{Beta}(\alpha + S_n,\ n - S_n + \beta)$$

under squared loss Bayes Estimator $\hat{p}$ is
The posterior mean $\hat{p} = \int p\, f(p \mid x^n)\, dp$

Mean of $B(\alpha + S_n,\ n - S_n + \beta)$

$$\hat{p} = \frac{\alpha + S_n}{\alpha + \beta + n}$$

ii)

$$R(\hat{p}, p) = [E[\hat{p}] - p]^2 - \text{var}(\hat{p})$$

$$= \left(\frac{\alpha + np}{\alpha + \beta + n} - p\right)^2 - \frac{np(1-p)}{(\alpha+\beta+n)^2}$$

$$= \frac{[\alpha + np - p(\alpha + \beta + n)]^2 - np + np^2}{(\alpha+\beta+n)^2}$$

$$= \frac{\alpha^2 - 2\alpha(\alpha+\beta)p + p^2(\alpha+\beta)^2 - np + np^2}{(\alpha+\beta+n)^2}$$

$$R(\hat{p}, p) = \frac{\alpha^2 - p\left(2\alpha(\alpha+\beta) - n\right) + p^2\left[(\alpha+\beta)^2 + n\right]}{(\alpha+\beta+n)^2}$$

$$= \left(\frac{\alpha}{\alpha+\beta+n}\right)^2 - 2p\,\frac{\left(\alpha(\alpha+\beta)-n\right)}{(\alpha+\beta+n)^2} + p^2\left[---\right]$$

parabola

$$\sup_p R(\hat{p}, i) = \infty.$$

can't find minimax like this.

iii) $r = \int R(\hat{p}, p)\,\pi(p)\,dp.$

$\pi(x) = \frac{\int \theta\, L_n(\theta)\, \pi(\theta)}{\int L_n(\theta)\pi(\theta)}$

we need to choose α, β so p disaper to get a constant risk estimator $\Rightarrow$ MiniMax.

$$\begin{cases} n = 2\alpha(\alpha+\beta) \\ n = -(\alpha+\beta)^2 \end{cases}$$

$\frac{\sqrt{n}}{2} = \alpha = \beta$

$\cancel{2\alpha(\alpha+\beta) = -(\alpha+\beta)^2}$

$\cancel{(\alpha+\beta)(\alpha+\beta+2\alpha) = 0}$

$\cancel{(\alpha+\beta)(3\alpha+\beta) = 0.}$

$\cancel{\alpha = -\beta}$

$\cancel{\alpha = -\beta/3}$

Exercise MLE Normal.

$x_1, \dots x_n \sim N(\mu, \sigma^2)$

$$L_n(\mu) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp -\frac{1}{2\sigma^2}\sum_1^n (x_i-\mu)^2$$

$$\log L_n(\mu) = \log\left(\frac{1}{2\pi\sigma^2}\right)^{n/2} - \frac{1}{2\sigma^2}\sum_i^n (x_i-\mu)^2$$

$$\frac{\partial \log L_n(\mu)}{\partial\mu} = \frac{1}{\sigma^2 2}\sum_i^n (x_i-\mu) = \frac{1}{L_n(\mu)}\frac{\partial L_n(\mu)}{\partial\mu}.$$

equal to zero

$$\sum (x_i-\mu) = 0 \qquad \sum x_i - n\mu = 0$$

$$\mu = \frac{1}{n}\sum_i^n x_i$$

$$\mu = \bar{x}.$$

Rewrite

$$\sum (x_i-\mu)^2 =$$

$$\sum x_i^2 + \mu^2 - 2x_i\mu$$

$$= n\mu^2 - 2\mu n\bar{x} + \sum$$

complete squares

$$= n\left[\mu^2 - 2\mu\bar{x} + \bar{x}^2\right] - n\bar{x}^2 + \sum$$

$$= n(\mu-\bar{x})^2 + n\underbrace{\left[\frac{1}{n}\sum x_i^2 - \bar{x}\right.}_{\sigma^2}$$

$$= n(\mu-\bar{x})^2 + n\sigma^2$$

$$L_n(\mu) = \frac{e^{\frac{n}{2}}}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\mu-\bar{x})^2}$$

$$\frac{\cancel{\partial \log} L_n(\sigma^2)}{\cancel{\partial\sigma^2}} = \frac{1}{(\sqrt{2\pi})^n(\sigma^2)^{n/2}} \exp -\frac{1}{2\sigma^2}\sum^n (x_i-\mu)^2$$

$$\log L_n(\sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum^n (x_i-\mu)^2$$

$$\frac{\partial \log L_n\sigma^2}{\partial\sigma^2} = -\frac{n}{2}\frac{\cancel{2\pi}}{\cancel{2\pi}\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum (x_i-\mu)^2 = 0$$

$$\sigma^2 = \frac{1}{n}\sum (x_i-\mu)^2.$$

Exercise. Posterior Gaussian

$$\pi(\mu \mid x^n) = \frac{f(x^n \mid \mu)\,\pi(\mu)}{m(x)}$$

$$L_n f(x^n \mid \mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp -\frac{1}{2}\sum (x_i-\mu)^2 = \frac{e^{n/2}}{\sqrt{2\pi\sigma^2}} \exp -\frac{n}{2\sigma^2}(\bar{x}-\mu)^2$$

given $\pi(\mu) \propto \exp\left[-\frac{1}{2b^2}(\mu-a)^2\right]$

$$L_n(x^n \mid \mu)\,\pi(\mu) \propto \exp\left[-\frac{1}{2b^2}(\mu-a)^2 - \frac{n}{2\sigma^2}(\bar{x}-\mu)^2\right]$$

$$= \exp\left[-\frac{1}{2}\left(\frac{\sigma^2(\mu-a)^2 + b^2 n(\bar{x}-\mu)^2}{b^2\sigma^2}\right]\right.$$

$$= \exp\left[-\frac{1}{2}\left(\frac{\sigma^2(\mu^2+a^2-2\mu a) + nb^2(\bar{x}^2+\mu^2-2\bar{x}\mu}{b^2\sigma^2}\right.\right.$$

$$= \exp -\frac{1}{2}\frac{\left[\mu^2(\sigma^2+nb^2) - 2\mu(a\sigma^2+nb^2\bar{x}) + a^2\sigma^2 + nb^2\bar{x}^2\right]}{b^2\sigma^2}$$

$$= \exp -\frac{1}{2(b^2\sigma^2)}\left[\begin{array}{l}\mu^2(\sigma^2+nb^2) - 2\mu(a\sigma^2+nb^2\bar{x}) \\ \pm (a\sigma^2+nb^2\bar{x})^2 \\ + a^2\sigma^2+nb^2\bar{x}^2\end{array}\right]$$

$$\propto \exp -\frac{1}{2}\left\{\frac{1}{\frac{(b^2\sigma^2)}{\sigma^2+nb^2}}\left[\left[\mu - \left(\frac{a\sigma^2+nb^2\bar{x}}{\sigma^2+nb^2}\right)\right]^2\right.\right\}$$

$\mu_p = a\sigma^2+nb^2\bar{x} / \sigma^2+b^2n$

$\sigma_p^2 = b^2\sigma^2 / \sigma^2+nb^2$

$N(\mu_p, \sigma_p^2)$

$$f(\mu \mid x^n) = N(\mu_p, \sigma_p^2)$$

under square loss l. Bayes Estimator $\hat{\mu}$ is posterior mean because minimizing Bayes risk is equivalent to minimize posterior risk

$$\Rightarrow \frac{dr}{d\hat{\theta}} = \frac{d}{d\hat{\theta}} \int (\theta - \hat{\theta})^2 f(\theta \mid \mu)\, d\theta = 2 \int (\theta - \hat{\theta}) f(\theta \mid \mu)\, d\theta = 0.$$

$$\Rightarrow \hat{\theta} = \int \theta f(\theta \mid \mu)\, d\theta.$$

$$\hat{\mu} = \mu_p = \frac{nb^2 \bar{x} + \sigma^2 a}{\sigma^2 + nb^2} \quad \text{Bayes Estimato}$$

iii) Risk of $\hat{\mu}$

$$R(\hat{\mu}, \mu) = (\text{bias}(\hat{\mu}))^2 + \text{var}(\hat{\mu})$$

$$= [E(\hat{\mu}) - \mu]^2 + \text{var}(\hat{\mu})$$

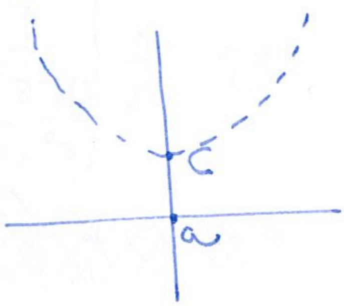
$$= \left[E\left[\frac{nb^2 \bar{X} + \sigma^2 a}{\sigma^2 + nb^2}\right] - \mu\right]^2 + \text{var}\left[\frac{nb^2 \bar{X} + \sigma^2 a}{\sigma^2 + nb^2}\right]$$

$$= \left[\frac{nb^2 \mu + \sigma^2 a}{\sigma^2 + nb^2} - \mu\right]^2 + \frac{(nb^2)^2 \frac{1}{n^2} n \sigma^2}{(\sigma^2 + nb^2)^2}$$

$$= \left[\frac{\cancel{nb^2 \mu} + \sigma^2 a - (\sigma^2 + \cancel{nb^2})\mu}{\sigma^2 + nb^2}\right]^2 + \frac{nb^4 \sigma^2}{(\sigma^2 + b^2 n)^2}$$

$$R(\hat{\mu}, \mu) = \left[\frac{\sigma^2 (a - \mu)}{\sigma^2 + nb^2}\right]^2 + \underbrace{\frac{nb^4 \sigma^2}{(\sigma^2 + b^2 n)^2}}_{c}$$

parabola on μ.



$$\sup_{\Theta} R(\hat{\mu}, \mu) = \infty$$

iii) Bayes Risk.

$$r = \int R(\hat{\mu}, \mu)\, \pi(\mu)\, d\mu$$

$$= \left(\frac{\sigma^2}{\sigma^2 + nb^2}\right)^2 \int (a-\mu)^2 \pi(\mu)\, d\mu + \frac{nb^4\sigma^2}{(\sigma^2 + b^2 n)^2} \int \pi(\mu)\, d\mu$$

$$\pi(\mu) \sim N(a, b^2)$$

$$r = \left(\frac{\sigma^2}{\sigma^2 + nb^2}\right)^2 b^2 + \frac{nb^4\sigma^2}{(\sigma^2 + b^2 n)^2}$$