Exercise: Risk Bernoulli iid $\quad \theta^{x}(1-\theta)^{1-x}$

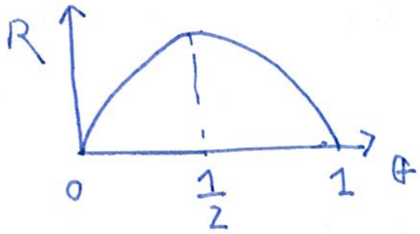
$\bar{X}_n$ is always unbiased

$$R(\bar{X}_n, \theta) = E(\bar{X}_n - \theta)^2 = var(\bar{X}_n) \overset{iid}{=} \frac{1}{n^2}\sum_{i=1}^{n} var(X) =$$

$$= \frac{1}{n^2}\, n\, var(X) = \frac{1}{n^2}\, n\, \theta(1-\theta) = \frac{\theta(1-\theta)}{n}$$

Bernoulli $EX = p \quad EX^2 = p$

$$\frac{dR}{d\theta} = \frac{d}{d\theta}\left(\frac{\theta - \theta^2}{n}\right) = \frac{1-2\theta}{n} = 0 \implies \theta_{max} = \frac{1}{2}$$



$$R_{max} = \frac{1}{4n}.$$

$$\implies var(\bar{X}_n) \leq \frac{1}{4n}.$$

chebyshev $\quad \mathbb{P}(|\bar{X}_n - \theta| > \epsilon) \leq \frac{1}{4n\epsilon^2}$

Exercise 2 $\quad R(\hat{\theta}_n, \theta) = var(\hat{\theta}_n) + bias^2(\theta_n)$

$$(\hat{\theta}_n - \theta)^2 = \left(\hat{\theta}_n - E(\hat{\theta}_n) + E(\hat{\theta}_n) - \theta\right)^2$$

$$= [\hat{\theta}_n - E(\hat{\theta}_n)]^2 + [\underbrace{E(\hat{\theta}_n) - \theta}_{\#}]^2 + 2(\hat{\theta}_n - E(\hat{\theta}_n))\underbrace{(E(\hat{\theta}_n) - \theta)}_{\#}$$

$$E(\hat{\theta}_n - \theta)^2 = E(\hat{\theta}_n - E(\hat{\theta}_n))^2 + \cancel{E}[E(\hat{\theta}_n) - \theta]^2$$

$$\cancel{+ 2(E(\hat{\theta}_n) - \theta)\, \underbrace{E[\hat{\theta}_n - E(\hat{\theta}_n)]}_{E(\hat{\theta}_n) - E(\hat{\theta}_n) = 0}}$$

$$= E(\hat{\theta}_n - E(\hat{\theta}_n))^2 + [E(\hat{\theta}_n) - \theta]^2$$

$$= var(\hat{\theta}_n) + [bias(\hat{\theta}_n)]^2$$

Exercise 3 Confidence Int Bernoulli

$$P(|\bar{X}_n - \theta| > t) \leq \frac{1}{4nt^2} = \alpha$$

$$t_\alpha = \sqrt{\frac{1}{4n\alpha}}$$

$$P(|\bar{X}_n - \theta| \leq t_\alpha) \geq 1 - \alpha$$

$$|\bar{X}_n - \theta| \leq t_\alpha \quad , \quad -t_\alpha \leq \bar{X}_n - \theta \leq t_\alpha$$

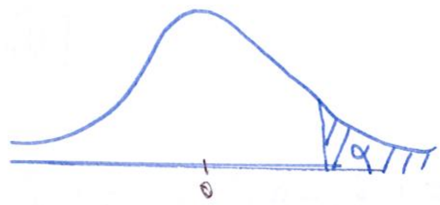
$$\bar{X}_n - t_\alpha \leq \theta \leq \bar{X}_n + t_\alpha$$

$$\Rightarrow \quad \bar{X}_n - \frac{1}{2\sqrt{n\alpha}} \leq \theta \leq \bar{X}_n + \frac{1}{2\sqrt{n\alpha}}$$

CLT $\quad P(\bar{X}_n - \theta \geq t_\alpha) = P\left(\sqrt{\frac{n}{\theta(1-\theta)}}(\bar{X}_n - \theta) \geq \frac{t_\alpha \sqrt{n}}{\sqrt{\theta(1-\theta)}}\right)$

from Table $\quad u_\alpha = \frac{t_\alpha \sqrt{n}}{\sqrt{\theta(1-\theta)}}$

$$P(Z_n \geq u_\alpha) \leq \alpha$$



$$P\left(\bar{X}_n - \theta \geq \sqrt{\frac{\theta(1-\theta)}{n}}\, u_\alpha\right) \leq \alpha$$

$$P\left(\bar{X}_n - \theta \geq \sqrt{\frac{1}{4n}}\, u_\alpha\right) \leq \alpha$$

$$P\left(|\bar{X}_n - \theta| \leq \frac{1}{2\sqrt{n}} u_{\frac{\alpha}{2}}\right) \geq 1 - \alpha$$

$$\bar{X}_n - \frac{1}{2\sqrt{n}} u_{\frac{\alpha}{2}} \leq \theta \leq \bar{X}_n + \frac{1}{2\sqrt{n}} u_{\frac{\alpha}{2}}$$

Theorem F continuous and differentiable at $q(u)$ s.t $F'(q(u)) > 0$

Then $\sqrt{n}\left(\hat{q}_n(u) - q(u)\right) \xrightarrow{\mathcal{L}} N\left(0, \frac{u(1-u)}{[F'(q(u))]^2}\right)$

Exercise Exponential $\frac{1}{\theta} e^{-x/\theta} \mathbb{1}_{x>0}$ $\begin{cases} var(X) = \theta^2 \\ E(X) = \theta. \end{cases}$

i) $E[X] = \int_0^\infty x \frac{1}{\theta} e^{-x/\theta} dx = x \frac{1}{\theta}(-\theta) e^{-x/\theta} \Big|_0^\infty + \int_0^\infty e^{-x/\theta} dx.$

$= \theta$

$F(x) = \int_{-\infty}^{x} \frac{1}{\theta} e^{-x/\theta} dx \, \mathbb{1}_{x>0} = \int_0^{x} \frac{1}{\theta} e^{-x/\theta} dx.$

$= \frac{1}{\theta}(-\theta) e^{-x/\theta} \Big|_0^x = -e^{-x/\theta} + 1$

$q(\frac{1}{2}) = m$ $\quad F(m) = \frac{1}{2}$ $\Rightarrow$ $1 - e^{-m/\theta} = \frac{1}{2}$

$e^{-m/\theta} = \frac{1}{2}$

$+\frac{m}{\theta} = \ln 2$ $\quad \hat{q}_n(\frac{1}{2}) = \hat{\theta}_n \ln 2$

$m = \theta \ln 2.$ $\quad \hat{m}_n = \ln 2 \, \hat{\theta}_n.$

2 Estimators of θ $\quad$ * $\hat{\theta}_n = \bar{X}_n$ $\quad$ * $\hat{\theta}_n = \frac{1}{\ln 2} X_{[n/2]}$

The Estimator $\hat{\theta}_n = \bar{X}_n$ $\qquad$ $var(x)$

$$\sqrt{n}\,(\bar{X}_n - \theta) \xrightarrow{\mathcal{L}} N(0, \theta^2)$$

the Estimator $\hat{q}_n(\tfrac{1}{2}) = \hat{m}_n$

Theorem $\quad \sqrt{n}\,(\hat{q}_n(u) - q(u)) \xrightarrow{\mathcal{L}} N\left(0, \dfrac{u(1-u)}{[F'(q(u))]^2}\right)$

$$\sqrt{n}\,(\hat{q}_n(\tfrac{1}{2}) - q(\tfrac{1}{2})) \xrightarrow{\mathcal{L}} N\left(0, \frac{\frac{1}{4}}{\frac{1}{4\theta^2}}\right) = N(0, \theta^2$$

$$[F'(q(\tfrac{1}{2}))]^2 = [F'(\theta \operatorname{Ln} 2)]^2 = \left(\frac{1}{\theta} e^{-\frac{\theta \operatorname{Ln} 2}{\theta}}\right)^2 = \left(\frac{1}{2\theta}\right)^2 = \frac{1}{4\theta^2}.$$

~~As~~ $\sqrt{n}\,(\hat{q}_n(\tfrac{1}{2}) - q(\tfrac{1}{2})) \xrightarrow{\mathcal{L}} N(0, \theta^2)$

$\hat{q}_n(\tfrac{1}{2}) = \hat{\theta}_n \operatorname{Ln} 2.$

$$\sqrt{n}\,(\hat{\theta}_n - \theta) = \frac{\sqrt{n}}{\operatorname{Ln} 2}\left(\hat{q}_n(\tfrac{1}{2}) - q(\tfrac{1}{2})\right)$$

$$\xrightarrow{\mathcal{L}} \frac{1}{} N(0, \theta^2) = N(0, \frac{\theta^2}{} \operatorname{Ln} 2^2)$$

Delta Method

$g(x) = \operatorname{Ln} 2\, x$ $\qquad \sqrt{n}\,(g(\hat{\theta}_n) - g(\theta)) \xrightarrow{\mathcal{L}} N(0, \theta^2 g'(\theta)^2)$

$g'(x) = \operatorname{Ln} 2$

$$N(0, \theta^2 (\operatorname{Ln} 2)^2)$$

Exercise Delta Method Bernoulli

$X_1, X_2, \dots X_n \sim B(\theta)$

$$\sqrt{n}\,(\hat{\theta}_n - \theta) \xrightarrow{L} N(0,1)\ \sigma.$$

$$\sqrt{n}\,(\hat{\theta}_n - \theta) \xrightarrow{L} \sqrt{\theta(1-\theta)}\ N(0,1)$$

$$\sqrt{n}\,(g(\hat{\theta}_n) - g(\theta)) \xrightarrow{L} g'(\theta)\ \sqrt{\theta(1-\theta)}\ N(0,1)$$

$$g'(\theta) = \frac{1}{\sqrt{\theta(1-\theta)}}$$

$$(-\theta^2+\theta) = -\theta^2 + \theta - \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2$$

$$= -\left(\theta - \tfrac{1}{2}\right)^2 + \tfrac{1}{4}$$

$$= \left[-4\left(\theta - \tfrac{1}{2}\right)^2 + 1\right] \tfrac{1}{4}.$$

$$f(f^{-1}(x)) = x$$

$$\frac{d}{dx} f(f^{-1}(x)) = 1$$

$$f'(f^{-1}(x)) \frac{df^{-1}}{dx} = 1$$

$$\frac{df^{-1}}{dx} = \frac{1}{f'(f^{-1}(x))}$$

$$g'(\theta) = \frac{1}{\sqrt{\frac{1}{4}\left(1 - 4\left(\theta - \frac{1}{2}\right)^2\right)}} =$$

$$= \frac{2}{\sqrt{1 - (2\theta - 1)^2}}$$

$u = 2\theta - 1$

$du = 2d\theta$

$$\int g'(\theta)\,d\theta = \int \frac{2}{\sqrt{1-u^2}} \frac{du}{2} =$$

$$= \arcsin u.$$

$$= \arcsin(2\theta - 1) = g(\theta)$$

Exercises

5-6) $E[\bar{X}] = E\left[\frac{X_1 + X_2 + \dots + X_n}{n}\right] = \frac{1}{n}\left[E[X_1] + E[X_2], \dots\right]$

same distribution $= \frac{1}{n} n \mu = \mu$

$$\underline{\underline{E[\bar{X}] = \mu}}.$$

5-7) $Var(\bar{X}) = Var\left(\frac{1}{n}\sum_i^n X_i\right) = \frac{1}{n^2} Var\left(\sum_i^n X_i\right)$

~~indep~~ independent $= \frac{1}{n^2}\left[var(X_1) + Var(X_2) + \dots + Var(X_n)\right]$

identical $= \frac{n}{n^2} Var(X_i) = \frac{1}{n}\sigma^2$

$$\underline{\underline{var(\bar{X}) = \frac{1}{n}\sigma^2}}$$

5-20) $X_j - \bar{X} = (X_j - \mu) - (\bar{X} - \mu)$

$(X_j - \bar{X})^2 = (X_j - \mu)^2 - 2(X_j - \mu)(\bar{X} - \mu) + (\bar{X} - \mu)^2$

$E\left[\frac{\sum_i^n (X_j - \bar{X})^2}{n}\right] = E\frac{1}{n}\sum_i^n\left[\cancel{E}[X_j - \mu]^2\right] - 2(\bar{X} - \mu)\frac{E}{n}\underbrace{\sum_i^n (X_j - \mu)}_{n\bar{X} - n\mu}$

$+ \underbrace{\frac{1}{n} E \sum_i^n (\bar{X} - \mu)^2}_{\frac{1}{n} nE(\bar{X} - \mu)}$

$\sigma^2\left(1 - \frac{1}{n}\right) = \sigma^2\left(\frac{n-1}{n}\right) = \sigma^2 - 2E(\bar{X} - \mu)^2 + E(\bar{X} - \mu)^2$ $\quad \underbrace{\phantom{E(\bar{X}-\mu)}}_{\frac{\sigma^2}{n}}$

$= \sigma^2 - \frac{\sigma^2}{n}$

Mean Squared Error

$$E[\hat{e} - \theta]^2 = E[\hat{\theta} - E(\hat{\theta}) + (\theta - E(\hat{\theta}))]^2$$

$$= E\left[(\hat{\theta} - E(\hat{\theta}))^2 + (\theta - E(\hat{\theta}))^2 - 2(\hat{\theta} - E(\hat{\theta}))(\theta - E(\hat{\theta}))\right]$$

$$= E[\hat{e} - E(\hat{\theta})]^2 + E[\theta - E(\hat{\theta})]^2 - 2E\left[(\hat{\theta} - E(\hat{\theta}))(\underset{\#}{\theta} - \underset{\#}{E(\hat{\theta})})\right]$$

$$= E[\hat{e} - E(\hat{\theta})]^2 + E[\theta - E(\hat{\theta})]^2 - 2(\theta - E(\hat{\theta}))\underbrace{E[\hat{\theta} - E(\hat{\theta})]}_{\substack{E[\hat{\theta}] - E\,E(\hat{\theta}) \\ = 0}}$$

$$= E[\hat{\theta} - E(\hat{\theta})]^2 + E[\theta - E(\hat{\theta})]^2$$

$$= E[\hat{e} - E(\hat{\theta})]^2 + (\theta - E(\hat{\theta}))^2$$

$$= \operatorname{var}(\hat{e}) + \operatorname{Bias}^2(\hat{\theta})$$

Exercises

$$\mathbb{P}(\theta \in C_n) = \mathbb{P}\left(\hat{\theta} - z_{\frac{\alpha}{2}} \hat{se} < \theta < \hat{\theta} + z_{\frac{\alpha}{2}} \hat{se}\right)$$

$$= \mathbb{P}\left(-z_{\frac{\alpha}{2}} < \frac{\theta - \hat{\theta}}{\hat{se}} < z_{\frac{\alpha}{2}}\right)$$

$$= \mathbb{P}\left(-z_{\frac{\alpha}{2}} < Z < z_{\frac{\alpha}{2}}\right)$$

$$= 1 - 2\frac{\alpha}{2} = 1 - \alpha.$$

Example 7.15 Wasserman

$X_i \sim Ber(p)$

$$\hat{p} = \bar{X} = \frac{1}{n}\sum X_i \quad , \quad Var(\hat{p}) = Var\left(\frac{1}{n}\sum X_i\right)$$

$$= \frac{1}{n^2}\sum^{n} Var(X_i)$$

$$= \frac{1}{n^2}\sum^{n} p(1-p)$$

$$= \frac{n}{n^2} p(1-p) = \frac{p(1-p)}{n}$$

By CLT $\hat{p} \xrightarrow{L} N(p, \hat{se}^2)$

$$\frac{\hat{p} - p}{\hat{se}} \xrightarrow{L} N(0,1) \qquad \hat{se}^2 = \frac{\sigma^2}{n}.$$

$\Rightarrow$ $1-\alpha$ C is $\hat{p} \pm z_{\frac{\alpha}{2}} \hat{se}$

$$\frac{1}{n}\sum X_j - \bar{X} = S$$

$$X_j - \bar{X} = (X_j - m) - (\bar{X} - m)$$

$$(X_j - \bar{X})^2 = (X_j - m)^2 + (\bar{X} - m)^2 - 2(\bar{X} - m)(X_j - m)$$

$$\sum (X_j - \bar{X})^2 = \sum (X_j - m)^2 + \sum (\bar{X} - m)^2 - 2(\bar{X} - m)\underbrace{\sum (X_j - m)}_{n(\bar{X} - m)}$$

$$= n\sigma^2 + n(\bar{X} - m)^2 - 2n(\bar{X} - m)^2$$

$$\sum (X_j - \bar{X})^2 = n\sigma^2 - n(\bar{X} - m)^2$$

$$E\ \frac{1}{n}(X_j - \bar{X})^2 = \sigma^2 - \underbrace{E\left[(\bar{X} - m)^2\right]}_{\frac{\sigma^2}{n}}$$

$$= \sigma^2\left(1 - \frac{1}{n}\right) = \sigma^2\left(\frac{n-1}{n}\right)$$

$$E\left[\frac{\sum X_j - \bar{X}}{n}\right] = \sigma^2\left(\frac{n-1}{n}\right)$$

$$S = \frac{1}{n}\sum (X_i - X_n)^2 = \frac{1}{n}\sum^n X_i^2 - \frac{2}{n}\sum^n X_i \bar{X}_n + \underbrace{\frac{1}{n}\sum \bar{X}_n^2}_{\bar{X}_n^2}$$

$$= \frac{1}{n}\sum X_i^2 - 2\bar{X}_n\bar{X}_n$$

$$\frac{1}{n}\sum X_i^2 - \bar{X}_n^2$$

Linear Gaussian Model

$\epsilon_i \sim N(0,1)$

$$Y = X\beta + \sigma\epsilon \qquad X = \begin{pmatrix} x_{11} & \cdots & x_{1d} \\ \vdots & & \\ x_{n1} & \cdots & x_{nd} \end{pmatrix} \qquad \epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_n \end{pmatrix}$$

$$\hat{\beta} = (X^T X)^{-1} X^T Y$$

$$\frac{1}{\sigma^2} \sum^{n} \epsilon_i^2 = \frac{1}{\sigma^2} \sum^{n} (Y - X\hat{\beta})_i^2 =$$

$$\sum^{n} [N(0,1)]^2 \sim \chi^2_{(n)} \quad ; \quad \frac{N(0,1)}{\sqrt{\dfrac{\chi^2_{(n-d)}}{(n-d)}}} \sim t_{(n-d)}$$

$$\frac{\dfrac{\chi^2_{(d_1)}}{d_1}}{\dfrac{\chi^2_{(d_2)}}{d_2}} \sim F(d_1, d_2)$$

Example

$$Y_1, Y_2, \dots Y_n \sim N(m, \sigma^2)$$

$$Y = X\beta + \sigma\epsilon$$

$$Y = (1, 1, 1 \dots)^T m + \sigma\epsilon \qquad X = (1, 1, 1, 1)^T$$

$$X^T X = (1, 1, 1 \dots) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = n$$

$$\hat{\beta} = (X^T X)^{-1} X^T Y = \bar{y}_n \qquad X\beta = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{y}_n = \begin{pmatrix} \bar{y}_n \\ \bar{y}_n \\ \vdots \end{pmatrix}$$

$$\frac{1}{\sigma^2} \sum^{n} \epsilon_i = \underbrace{\frac{1}{\sigma^2} \sum^{n} (\bar{y}_n - Y_i)^2}_{\chi^2_{(n-d)}} = (n-d) \frac{\hat{\sigma}^2}{\sigma^2} \qquad \hat{\sigma}^2 = \frac{1}{(n-d)} \sum^{n} (\bar{y}_n - Y_i)^2$$

$$(n-d) \frac{\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{(n-d)} \qquad \frac{1}{\sigma^2} \sum^{n} (m - Y)^2 \sim \chi^2_{(n)}$$

Because $\hat{\beta} = \bar{Y}_n = \frac{1}{n}\sum^{n} Y_i \sim N(m, \frac{\sigma^2}{n})$

$$var\left(\frac{1}{n}\sum Y_i\right) = \frac{n}{n^2} var(Y) = \frac{\sigma^2}{n}.$$

$$\hat{\beta} \sim N(m, \frac{\sigma^2}{n})$$

$\frac{\sqrt{n}}{\sigma}(\hat{\beta} - m) \sim N(0,1)$ but σ is unknown.

$$(n-d)\frac{\hat{\sigma}^2}{\sigma^2} \longrightarrow \chi^2_{(n-d)}$$

$$\frac{\sqrt{n}}{\hat{\sigma}}(\hat{\beta} - m) = \frac{\sqrt{n}\,(\hat{\beta} - m)}{\sigma\sqrt{\frac{\chi^2_{(n-d)}}{(n-d)}}} = \frac{N(0,1)}{\sqrt{\frac{\chi^2_{(n-d)}}{n-d}}} \sim t_{(n-d)}.$$

$$\frac{\sqrt{n}}{\hat{\sigma}}(\hat{\beta} - m) \sim t_{(n-d)}.$$

Example 2.

$$Y_i = \beta_0 + \beta_1 x_i + \sigma \epsilon_i$$

$$= \beta_0 + \beta_1 (x_i - \bar{x}_n + \bar{x}_n) + \sigma \epsilon_i$$

$$= (\beta_0 + \beta_1 \bar{x}_n) + \beta_1 (x_i - \bar{x}_n) + \sigma \epsilon_i$$

The new $x_i' = x_i - \bar{x}_n$

$$\sum x_i' = 0.$$

$$Y_i = \beta_0 + \beta_1 x_i' + \sigma \epsilon_i$$

$$= X'\beta + \sigma \epsilon.$$

$$= \begin{pmatrix} 1 & \\ 1 & X \\ 1 & \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + \sigma \epsilon \qquad X' = \begin{pmatrix} x_1 - \bar{x}_n \\ \vdots \\ x_2 - \bar{x}_n \end{pmatrix}$$

obvious $\mathbb{1} \perp X$ so rank $= 2$.

$$X'^T X' = \begin{pmatrix} 1 & 1 & \cdots \\ x_1 & x_2 & \cdots \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & \vdots \end{pmatrix} = \begin{pmatrix} n & \sum x_i' \\ \sum x_i' & \sum x_i'^2 \end{pmatrix}$$

$$X^T X = \begin{pmatrix} n & 0 \\ 0 & \sum x_i'^2 \end{pmatrix} \qquad (X^T X)^{-1} = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{\sum x_i'^2} \end{pmatrix}$$

$$\hat{\beta} = (X^T X)^{-1} X^T Y = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{\sum x_i'^2} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots \\ x_1 & x_2 & \cdots \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{\sum x_i'^2} \end{pmatrix} \begin{pmatrix} \sum Y_n \\ \sum x_i' Y_i \end{pmatrix} = \begin{pmatrix} \bar{Y}_n \\ \frac{\sum x_i' Y_i}{\sum x_i'^2} \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

$$\hat{\beta}_1 = \frac{\sum X_i' Y_i}{\sum X_i'^2} = \frac{\sum (X_i - \bar{X}_n) Y_i}{\sum (X_i - \bar{X}_n)^2}$$

$$= \frac{\sum X_i Y_i - \sum Y_i \bar{X}_n}{n \hat{\mathrm{var}}(X)}$$

$$= \frac{\hat{\mathrm{cov}}(X, Y)}{\hat{\mathrm{var}}(X)}.$$

$$\hat{\beta}_0 = \bar{Y}_n$$

$$\hat{\beta} \sim N\left(\overset{m}{\beta}, \frac{\sigma^2}{X^T X}\right)$$

$$\hat{\beta}_1 = \frac{\sum X_i' Y_i}{\sum X_i'^2} = \beta + \frac{\sum \sigma \epsilon X_i'}{\sum X_i'^2}$$

$$\hat{\sigma}^2 = \frac{1}{(n-p)} \sum^n (\bar{Y}_n - Y)^2 \qquad (n-d)\frac{\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{(n-d)}$$

$$\frac{\sqrt{n}(\hat{\beta} - \beta)}{\hat{\sigma}} \sim T_{(n-d)}$$