How Hilbert has found the Einstein Equations before Einstein and forgeries of Hilbert's page proofs

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Abstract: A succinct chronology is given around Nov 1915, when the explicit field equations of General Relativity have been found. Evidence, unearthed by D.Wuensch, that a decisive document of Hilbert has been mutilated in recent years with the intention to distort the historical truth is reviewed and discussed. The procedure how Hilbert has found before Einstein the correct equations "easily without calculation" by invariant-theoretical arguments is identified for the first time. However, Hilbert has based his derivation on an incorrect or at least not yet formally proved invariant theoretical fact.

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1 Introduction

At Nov 11, 1915 Einstein still submits incorrect field-equations (2) of General Relativity, lacking the trace term, to *Preußische Akademie der Wissenschaften zu Berlin*^[1]. At Nov 20, for the first time, David Hilbert submits the correct equations (5c) to *Gesellschaft der Wissenschaften zu* $Göttingen^{[2]}$. Einstein, 5 days later, now also submits the correct equations (7) at Berlin^[3], but without citing Hilbert, although with a postcard^[4] dated Nov 18, Einstein acknowledges receipt of a postcard from Hilbert (probably dated Nov 16) containing field equations for General Relativity. Hilbert's postcard is lost.

Most physicist, including those working on General Relativity, either have no definite opinion or they believe that General Relativity was the creation of Einstein alone. Only a few, interested in history of science, believed that Hilbert has first published the correct equations, and Einstein 5 days later, either independently or through the influence of Hilbert, arrived also at the correct equations.

In 1997 this opinion was challenged by L.Corry, J.Renn and J.Stachel (CRS). After having found in a Göttingen archive the page proofs^[2] of Hilbert's Nov 20 submission, which he has received from the printery at Dec 6, they publish in *Science* an article^[5] stating: *nowhere* in the whole page proofs the correct equations can be found. Thus they revert the opinion held thus far by historians, by insinuating Hilbert has copied from Einstein, and not Einstein from Hilbert, because Hilbert had added the explicit equations (5c) with the correct trace term only after Dec 6, appearing in his published version^[6] at Mar 31, 1916 only. Hilbert has made other considerable changes for his published version without altering the initial date Nov 20 of submission or adding a date of revision.

After 1997, a tremendous fact was unearthed, turning the Einstein/Hilbert priority dispute from a purely academic controversy among historians into a criminal story: In 2002 Winterberg discovers that about one third of a page in Hilbert's page proofs have been cut off, and Winterberg was the first to publish^[7] the opinion the cut-off is the deed of a forger with the intention to distort the historical truth in favour of Einstein. (For a list of earlier mentioning of the cut-off see Sommer^[8].)

CRS defend with vehemence their opinion^[9] Hilbert himself had made the cut-off. However, the most miraculous item in this case would then be the fact, that CRS did not mention the cut-off in

their *Science* article, though their main statement was that *nowhere* in the whole page proofs the explicit equations can be found.

The refereeing process in *Science* is very strict. Therefore we conclude that their referees have been in possession of a copy of the intact page proofs. Otherwise the referees had observed the cut-off and they had not passed the article in the present form. Similarly we conclude that indeed the intact page proofs did not contain the explicit field equations.

For historians it was a great triumph^[10] to have reverted the opinion about such a milestone in the development of physics as is the discovery of the correct explicit field equations of General Relativity. However, their triumph was shattered by the early argument Hilbert was the first to have identified the Ricci scalar R as the correct Lagrangian density and the elaboration by well known variational procedures were only a trivial exercise, thus Hilbert still deserved the priority. Thus a forger went out and made the cut-off in Hilbert's page proofs. In the remaining page proofs it can no longer be seen that Hilbert's K is indeed the 'invariant stemming from Riemann's tensor', i.e. the Ricci-scalar R, and Hilbert could have added that statement only for his published version after he has seen the correct field equations in Einstein's publications.

Hilbert's postcard (dated Nov 16 ± 1 day) in which he has communicated the correct explicit field equations to Einstein is lost, which is curious since more trivial postcards before and later are preserved. Only from Einstein's reactions we can safely infer that Hilbert's Nov 16 postcard indeed contained the correct field equations.

Hilbert's case was also weak because Hilbert did not give detailed explanations how he has found the equations, but says they follow 'easily without calculation' only mentioning invariant theoretical arguments. A straightforward elaboration of the variational procedure, without using any tricks discovered only later, though certainly possible, would require several days of calculation. I doubt if anyone has really done this 'great calculation', as I call it in the following, and it seems at least nobody has published it.

In the published version of his page proofs, Hilbert only gives the following explanations:

The first term on the left hand side of [our Eq.(5c)] follows easily without calculation from the fact that $K_{\mu\nu}$, except $g_{\mu\nu}$, is the only second rank tensor, and K is the only invariant which can be formed from $g^{\mu\nu}$ and its first and second derivatives $g_k^{\mu\nu}, g_{kl}^{\mu\nu}$. (1)

(Note that Hilbert writes K for the Ricci scalar R, $K_{\mu\nu}$ for the Ricci tensor $R_{\mu\nu}$, and he adds additional lower indices for partial derivatives with respect to the coordinates.)

I have pondered about Hilbert's 'easy' method, since I was dissatisfied by explanations given by others^[11], who have proposed simple but tricky procedures, and probably are methods discovered only later. Coincidentally I have ordered further copies of Hilbert's private notes from the Göttingen archive^[12], and I was puzzled about Hilbert mentioning, among general formulae he has obtained for Riemannian geometry, a 'special case' which seemed completely absurd for me until I realized that this 'special case' is sufficient to derive the Einstein Equations. For a skilled theorist, as Hilbert was, that calculation required perhaps 5-10 minutes, thus he described it as 'easily without calculation' meaning avoiding the straightforward but lengthy 'great calculation', which would require several days. Hilbert's mentioning that 'special case' in his private notices is a hint he actually proceeded along these lines. A lot of other special cases would do also. Therefore Hilbert did not specify one in his published version. In the following subsection 6.3 we give an even more elegant derivation, deserving the qualification 'easily without calculation'. It is based on a tacit assumption, which Hilbert was allowed using his famous power of mathematical intuition.

Unfortunately for Hilbert, the fact mentioned in (1), as it is stated, is incorrect, even if one assumes that with 'tensor' Hilbert means 'symmetric tensor', which perhaps was his terminology. Obviously Hilbert did not know the results of $\text{Haskins}^{[13]}$ and Zorawski^[14], though in 1908 the "Leipziger Berichte" were available at Göttingen. According to their results, see the following section (7.1),

for n = 4, including the Ricci scalar and Ricci tensor, there are 14 independent invariants and three tensors. Fortunately for Hilbert, the fact mentioned in (1) is correct if invariant and tensor are required to be linear in the second derivatives $g_{kl}^{\mu\nu}$. That modified statement (1) suffices for Hilbert's derivation.

However, that modified fact (1) was not known in 1915, and so Hilbert must have been aware his derivation of the explicit field equations is based on a shaky invariant theoretical fact, at least lacking a formal proof. This may explain why Hilbert did not include them in his page proofs. Also because for him, as a mathematician, explicit field equations did not seem to be of utmost importance. Only after he has seen them in Einstein's Nov 25 publication (without acknowledgment of his private Nov 16 postcard communication) he has added them in a hurry, together with the insufficient explanation (1).

In his republication^[15] 1924, i.e. almost ten years later, Hilbert writes in the introduction: "The following is essentially a reprint of my older communications ... with only slight editorial changes and rearrangements, which should improve its comprehension." Despite that statement, he abandons his previous derivation based on (1) completely and switches to the standard derivation, which in the meantime was found by others, using Riemannian normal coordinates. In the meantime Vermeil^[16], Weyl^[17] and Cartan^[18] have given independent formal proofs of Hilbert's modified fact. I do not see any reason for a republication, except Hilbert has realized his 1915 sketch of a derivation of the field equations based on (1) was incomplete. His introductory words should hide his own derivation was a guess only and not a formal proof.

2 A Chronology^[19]

Since **1907** Einstein was working toward a unification of special relativity with gravitation. He had the ingenious idea that gravitation has something to do with curvature of space-time, thus unifying geometry and gravitation, or geometrizing the gravitational force. The metrical tensor g_{ab} (depending upon chosen generalized coordinates x^k) is the formulation of the metric and of the gravitational field, replacing Cartesian coordinates and Newton's gravitational potential. Masses (equivalent to energy according to $E = mc^2$) are the source of curvature.

1913 M. Grossmann^[20], for the first time, proposed the Ricci-tensor R_{ab} as a covariant candidate expressing curvature in a gravitation theory as intended by Einstein. Unfortunately, Grossmann erroneously concluded that equations with the Ricci tensor cannot reproduce Newton's theory of gravitation in the limit of weak gravitational fields. According to the published version^[20], the error is due to Grossmann. However, in his Nov 4 paper (see below), Einstein takes the full responsibility for the error.

1914 and the first half of 1915: Tragically, based on this misjudgement by Grossmann, Einstein publishes several papers stating the gravitational field equations in the form of the following Eq. (2), but with the left-hand side not the Ricci-tensor, but several curious mathematical expressions, which are not generally covariant but only covariant under linear transformations.

(On the right hand side of (2), κ is the constant of gravitation in certain units, and T_{ab} is the energy-momentum tensor of matter or radiation, producing the curvature. T_{00} is the mass density. The other components are pressure, stress, and mass flux density, which also produce curvature.)

General covariance of a field equation such as (2) is mandatory for General Relativity, which is a theory of the metrical field. General covariance means that arbitrary coordinates might be chosen and the field equations such as (2) have the same form (covariance = form invariance), or in other words (2) should be valid in any coordinate system. Special coordinates can be defined using the metric. Thus special coordinates bear the danger that they restrict the generality of the gravitational field. June 28 - July 5, 1915 By Hilbert's invitation, Einstein gives six two-hour lectures about General Relativity in Göttingen.

Nov. 4, 1915 For the first time, $Einstein^{[1]}$ postulates the field equations in the form (2) with the left-hand side the Ricci-tensor. But unnecessarily, he uses a special expression for the Ricci tensor, which is valid only in special coordinates satisfying (4), where g is the determinant of the metrical tensor, so *manifest* general covariance is not yet reached.

The most important progress in that paper was that Einstein proved that (2) has Newton's gravity as a limit, thus correcting Grossmann's error, which had cost Einstein almost 3 years of futile work. Einstein writes^[1]: "Thus I regained general covariance, a requirement which I had abandoned with a heavy heart three years ago, while working with my friend Grossmann."

At the same time, Einstein freed himself from another quarrel he has had against his theory, namely his earlier observation that all equations of the form (2), with the left hand side having some (nontrivial, i.e. local) covariance, violate causality: Knowledge of the metrical tensor field in the past cannot predict it uniquely in the future with the help of (2). Einstein is now recognizing that this is because of covariance: Coordinates have no objective physical meaning and our free choice in the future (permitted by general covariance) cannot be predicted by a physical law. Nevertheless, coordinates are necessary to formulate the metrical-gravitational field, namely with the help of a metrical tensor g_{ab} . Therefore (2) must and can be supplemented by 4 additional conditions, nowadays called gauge-conditions, specifying the choice of coordinates in the future, thus restoring causality of the field equations of General Relativity.

Nov 7, 1915 Einstein sends Hilbert the page proofs of the above Nov 4 paper.

Nov 11, 1915 For the first time, Einstein^[1] proposes (manifestly) fully covariant field equations for the gravitational field:

$$R_{ab} = -\kappa T_{ab} \tag{2}$$

where R_{ab} is now the Ricci tensor formulated in arbitrary coordinates. (The minus sign on the right-hand side of (2) is conventional, depending on a conventional minus sign in the definition of the Ricci-tensor.) A milestone in the development of General Relativity was hereby achieved: (2) has Newton's theory of gravitation as a limit, though with κ half the value as given by (8), and (2) are (manifestly) generally covariant field equations, and most spectacularly (see Einstein's Nov 18 paper) (2) explains Mercury's perihelion precession, and (2) explains the deflection of light as (allegedly) observed at the 1919 eclipse of the sun by Eddington and Fröhlich.

Nevertheless Einstein was deeply dissatisfied with (2). As is now also the accepted view, Einstein was convinced that to every physical equation in special relativity (i.e. in the absence of a gravitational field, e.g. conservation of energy and momentum, Maxwell's equations, etc) there corresponds an analogous equation in General Relativity (i.e. in the presence of a gravitational field) whereby the partial derivatives with respect to the coordinates are replaced by covariant derivatives. Nowadays this is called the *principle of minimal coupling*, and is the accepted method for postulating Maxwell's equations in General Relativity. Conservation of energy and momentum in flat Minkowski space of special relativity thus leads to the equation

$$T_{a;b}^b = 0 \tag{3}$$

where ; denotes covariant derivative. Einstein observed that (2) does not satisfy (3) except in the case the trace $T = T_a^a$ vanishes. As we know today, the discrepancy between (2) and (3) is serious because applied to a point particle (e.g. the earth), (3) is equivalent to geodesic motion. Thus the empirical success of (2) is spoilt, because being based on the assumption of geodesic motion in the Schwarzschild metric.

Nov 12, 1915 Einstein writes to Hilbert: "Meanwhile, the problem has been brought one step

forward. Namely, the postulate

$$\sqrt{-g} = 1 \tag{4}$$

enforces general covariance." In modern terms, this means that the gauge-condition (4) (nowadays called Einstein gauge) is a *permitted* gauge, i.e. it does not restrict the generality of the gravitational field. More specifically: Using completely arbitrary coordinates in the most general gravitational field, it is always possible to introduce new coordinates fulfilling the gauge condition (4). In other words, Einstein's Nov 4 paper was already generally covariant, if not manifestly so in the physical sense that in every gravitational field one can choose coordinates fulfilling (4) and in these coordinates gravitation is governed by (2) in his Nov 4 version. The only physical reason for the requirement of general covariance is: not to restrict the generality of the gravitational field. *Manifest* general covariance is a matter of predilection. It has the advantage of making the non-restriction obvious i.e. manifest. With sufficient technical skill it is always possible to find the manifestly generally covariant formulation, the physical contents of the theory being unchanged.

Nov 13, 1915 With a postcard, Hilbert invites Einstein to attend his lecture at Göttingen scheduled for Nov 16, where he promises to give the solution of Einstein's 'great problem'. With 'great problem' Hilbert probably meant the state of affair when Einstein was lecturing in Göttingen in June/July. Obviously, he had not yet absorbed Einstein's Nov 4 page proofs, and had not realized, Einstein himself had already solved most of his 'great problem'. (Only the discrepancy with the requirement (3) remained.) Hilbert has brought this postcard to the post office in the night only, since it bears a postmark of Nov 14, 6-7 a.m.

Nov 13, 1915 After having posted the above postcard, Hilbert writes a second postcard, which again has received a postmark of Nov 14, 6-7 a.m. Obviously Hilbert has now studied Einstein's Nov 4 submission. Hilbert writes on this second postcard:

"As far as I understand your recent work, the solution given by you is completely different from mine, in particular since with me [the energy expression] necessarily contains the electric potential. Continued on page I with the invitation to come to here at Tuesday 6 p.m." With the Roman numeral 'I', Hilbert refers to his first postcard.

Nov 15, 1915 With a postcard, Einstein declines Hilbert's invitation for the Nov 16 talk because of fatigue and stomach-ache, and expresses his hope later to read the printed version of the talk.

Nov 16, 1915 Hilbert gives his talk at the Mathematical Academy of Göttingen.

Nov 16, 1915 Hilbert writes Einstein a postcard, which is now lost. So we do not know the exact date, which could vary by one day, nor exactly what was written on it. According to Wuensch^[21] the postcard contained the following three formulae

$$H = K + L \tag{5a}$$

$$\delta \int (K+L)\sqrt{g} \, dw = 0 \tag{5b}$$

$$\sqrt{g}\left(K_{\mu\nu} - \frac{1}{2}Kg_{\mu\nu}\right) + \frac{\partial L\sqrt{g}}{\partial g^{\mu\nu}} = 0$$
(5c)

(5c) are the correct field equations of General Relativity, including the trace term $-\frac{1}{2}Kg_{\mu\nu}$. The minus sign under the square root is absent giving the imaginary unit *i* which drops out of the equation. $K_{\mu\nu}$ is the Ricci tensor, in the tradition of Gauss for German Krümmung = curvature. So *K* is the Ricci scalar. (5b) is Hamilton's principle of least action. *H* is the Lagrangian, which by (5a) is the sum of the Lagrangian *K* of the gravitational field and the Lagrangian *L* of everything else (matter, radiation). Since in (5) Hilbert did not restrict *L* he still considers general matter (or radiation). Einstein's later reproach concerning Hilbert's treatment of matter in a letter to Ehrenfest^[22] as 'unnecessarily special' and in a letter to Weyl^[23] as 'childish' is thus completely unjustified. However, it is true, that later in his page proofs, Hilbert specialized *L* to the Lagrangian of Mie's electrodynamics, in which today nobody believes. Hilbert denotes the coordinates by w_a $(w = \text{world coordinates instead of } x^a \text{ as is usual today})$, so dw is the product of the four coordinate differentials in the fourfold integral in (5b), and $\sqrt{g} dw$ is the invariant volume element of spacetime. As is well known in General Relativity, given a Lagrangian L of matter (or radiation), its energy-momentum tensor is

$$T_{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\partial L \sqrt{g}}{\partial g^{\mu\nu}}.$$
(6)

The coupling constant κ does not occur in (5) as it can be put to unity while choosing suitable units.

It is curious that this Nov 16 postcard of Hilbert to Einstein is lost, while the more trivial Nov 13 and Nov 19 postcards are preserved.

Of course it is controversial what was on the postcard. A discussion will be found in the following section.

Nov 18, 1915 Einstein writes to Hilbert, acknowledging Hilbert's Nov 16 postcard:

"Dear Colleague! The system, given by you, coincides - as far I can see - exactly with what I have found in the last weeks and have sent to the Academy. The difficulty was not to find generally covariant equations for the $g_{\mu\nu}$; for this is easy with the help of the Riemann tensor. But it was difficult to recognize, that these equations are a generalization, and indeed a simple and natural one, of Newton's law. This I achieved only in the last weeks (I have sent you my first communication). The only possible generally covariant equations, which now have turned out to be the correct ones, I have taken into consideration with my friend Grossmann. Only with a heavy heart we parted with them, because the physical discussion has seemed me to have given the incompatibility with Newton's law. - Main point, the difficulties are now overcome. Today I send the Academy a paper, in which I deduce quantitatively from General Relativity, without any additional ad hoc hypotheses, Mercury's perihelion motion, as discovered by Leverrier. Thus far no other theory of gravitation has achieved that. With best greetings your Einstein."

The sentence 'I have sent you my first communication' clearly shows that Einstein refers to his Nov 4 paper he has sent to Hilbert at Nov 7, where (2) with the Ricci tensor is used. Probably he has found that equation several weeks ago, but has submitted it only at Nov 4. Thus in his first reaction, Einstein only recognized in (5c) that Hilbert has used the Ricci tensor. Probably only in the next few days, he observed Hilbert's additional trace term $-\frac{1}{2}Kg_{\mu\nu}$, and Einstein began to experiment with additional trace terms also.

The passage 'But it was difficult to recognize, that these equations are a generalization, and indeed a simple and natural one, of Newton's law.' is a heavy and very justified reproach against Hilbert. By good luck Hilbert has taken the Ricci scalar K as the Lagrangian of the gravitational field and by working out the corresponding Euler-Lagrange-equations (5c) (which of course is a tremendous achievement) for the first time has found the correct field equations of General Relativity. But because he did not show that these equations have Newton's theory as a limit, he does not know if his equations (5c) have possibly something to do with gravity. Hilbert had not even a change to do so because he specializes to Mie's electrodynamics and thus was not in a position to check if a point mass or a fluid moves according to Newton's gravity, as Einstein has done. It seems Hilbert did not even feel an obligation to do so.

Nov 18, 1915 Einstein submits his paper explaining Mercury's perihelion motion.

The biographer Pais^[24] writes: 'This discovery was, I believe, by far the strongest emotional experience in Einstein's scientific life, perhaps in all his life. Nature had spoken to him.' Einstein writes to Ehrenfest: 'For a few days I was beside myself with joyous excitement', and to Fokker that his discovery has given him palpitations of the heart.

Nov 19, 1915 On a postcard, Hilbert congratulates Einstein enthusiastically and in very friendly words for his explanation of Mercury's perihelion motion.

Nov 20, 1915 Hilbert submits his manuscript which was then printed on his page $\operatorname{proofs}^{[2]}$.

Nov 25, 1915 Einstein^[3] also arrives at the correct equations of General Relativity including the trace term:

$$R_{ab} = -\kappa \left(T_{ab} - \frac{1}{2} g_{ab} T \right) \tag{7}$$

at Berlin, but without citing Hilbert. To be in agreement with Newton's theory in the limit of weak fields one must have:

 $\kappa = 8\pi\gamma c^{-4} \tag{8}$

where γ is Newton's constant of gravitation, and c is the velocity of light. (7) is equivalent to (5c). Einstein does not give an explanation how he has now found the additional trace term. He has only checked that the conclusion T = 0 is no longer possible.

Nov 26, 1915 Einstein writes a letter to his friend Zangger accusing Hilbert, without naming him explicitly, in drastic words: "The theory has unique beauty. Only one colleague has understood it really, but he tries in a tricky way to 'nostrify' it (an expression due to Abraham). In my personal experience I have not learnt any better the wretchedness of the human species as on occasion of this theory and related to it. However, that does not concern me in the slightest." and in Nov 30, 1915 he writes Besso "Colleagues behave nastily."

It is difficult to understand such harsh words of Einstein. He must must have been extremely angry having worked for eight years to the solution of his 'great problem' and Hilbert in only a few weeks elegantly has found the solution before him. Einstein's fury shows the Hilbert's Nov 16 postcard was of considerable help to him.

On the other hand, Hilbert writes with great admiration about Einstein, e.g.^[15] "the publications of Einstein, which are always rich in new approaches and ideas."

Dec 2, 1915 Einstein's Nov 25 paper appears. Certainly Hilbert got a copy and very probably he was angry because Einstein did not credit him for his Nov 16 postcard with the correct equations (5c), and Hilbert has sent Einstein a letter expressing his grievance. Such a letter, which should have existed because of Einstein's mentioning of a 'certain resentment' at Dec 20, 1915, is also lost.

Dec 6, 1915 Hilbert receives his page proofs from the printery. Most probably Einstein received a copy from Hilbert in the following day.

Dec 20, 1915 Einstein asks Hilbert for reconciliation: "There has been a certain resentment between us, the cause of which I do not want to analyze any further. I have fought against the feeling of bitterness associated with it, and with complete success. I again think of you with undiminished kindness and I ask you to attempt the same with me. It is objectively a pity if two real guys that have somewhat liberated themselves from this shabby world are not giving pleasure to each other." In view of their friendly correspondence before, that offer for reconciliation is astonishing. Obviously other letters or postcards have been lost. Most probably Hilbert was angry Einstein did not reference him in the Nov 25 submission. However, Hilbert could not have uttered such a reproach outright, because then Einstein had nothing to analyze. Sommer^[8] suspects Einstein's offer was not an act of pure love. At that time Hilbert was much more famous than Einstein. A published reproach by Hilbert would have been disastrous for Einstein.

Mar 31, 1916 The printed version^[6] of Hilberts page proofs appear. He has made several revisions, retaining his date of submission, Nov 20, 1915, nor adding a date of revision.

March 7, 1918 Hilbert sends Klein his page proofs asking him to be careful to them sending them back because he had no other records.

1924 Hilbert's republication^[15], claiming 'rearrangements and editorial changes' only, but in fact

abandoning his own derivation and switching to one already standard at that time, using Riemannian normal coordinates.

1943 Hilbert's death.

1943-1967 Hilbert's documents are deposited in a adjoining room of the office of the Mathematical Institute at Göttingen^[21].

1966 About 130 letters written by Einstein, Planck, Born, Nernst, Debye, Sommerfeld, Weyl, Courant, Ehrenfest addressed to Hilbert have been sent to New York in order to copy them. They have been sent back but not deposited in the office again. Finally in the year 2000 they were found by Sommer^[8] on an attic of the widow of an assistant of Courant.

1967 The remaining documents (not having been sent to New York) came to the department of rare documents of the library of the university at Göttingen (called the 'Göttingen archive'^[12] in the following).

1985 The correspondence between Hilbert and Klein is published^[25].

1994 Corry detects Hilbert's page proofs at Göttingen. He orders copies which are sent to the Max Planck institute for the History of Science at Berlin where he studies them in the following year.

Nov 14, 1997 Science article by Corry, Renn and Stachel^[5], without mentioning the cut-off.

1999 Tilman Sauer^[26] for the first time mentions the cut-off. Also Renn and Stachel, but only in a footnote^[8].

2003 F. Winterberg^[7] for the first time publishes the suspicion the cut-off was the deed of a forger. Also in 2004 by C. J. Bjerknes^[27].

Mar 2005 Wuensch's book^[21] appears giving strong evidence the cut-off has been done in recent years, most probably by a skilled historian, because curious traces can be seen on Hilbert's page proofs, which find a plausible explanation only by attempts of cover-ups, presupposing knowledge of detailed historical facts.

3 What was on Hilbert's Nov 16, 1915 postcard to Einstein?

It is curious this postcard is lost, while his more trivial Nov 13 and Nov 19 postcards are preserved. However, we can safely infer what was essentially written on Hilbert's Nov 16 postcard, namely from Einstein's reactions:

Einstein Nov 18:

"The system, given by you, coincides - as far I can see - exactly with what I have found in the last weeks and have sent to the Academy. The difficulty was not to find generally covariant equations for the $g_{\mu\nu}$; for this is easy with the help of the Riemann tensor. But it was difficult to recognize, that these equations are a generalization, and indeed a simple and natural one, of Newton's law. This I achieved only in the last weeks."

This clearly demonstrates Hilbert has sent explicit field equations for general relativity (and e.g. not only a Lagrangian density).

The passage: "The system, given by you, coincides - as far I can see - exactly with what I have found in the last weeks and have sent to the Academy." opens the possibility Hilbert has sent incorrect field equations such as (2) lacking the trace term. However, this can be excluded by Einstein's angry reaction in his letter Nov 26 to his friend Zangger. Hilbert had had no chance to 'nostrify', if he had sent incorrect field equations.

Rather, in his first reaction, Einstein did not observe Hilbert's additional trace term but only Hilbert's use of the Ricci-tensor, so Einstein erroneously believed Hilbert's system coincided with his own.

Some historians believe Hilbert, after having seen the correct equations (7) in Einstein's paper, he had *copied* from Einstein when he has added the correct field equations for his final printed version somewhen after Dec 6. However, this can safely be excluded because Einstein did not give any method how he finally arrived at the correct equations (7), and in particular he did not derive them from a Lagrangian density. So it would be an extreme risk for Hilbert to *copy* from Einstein, since he could not know if Einstein's equations coincide with those following via variational calculus from his Lagrangian density.

4 How Einstein has found his equations

Einstein was dissatisfied with his equations (2), because they led to the unphysical conclusion T = 0. It is curious Einstein did not attempt a trace term on the left or right hand side of (2). It am sure, sooner or later, he alone or perhaps by a hint by someone less famous than Hilbert, had he attempted such trace terms. Perhaps, Einstein's Nov 26, 1915 letter to Zangger can be explained by his anger he has not come to this simple idea by his own.

4.1 Was Grossmann a poor mathematician?

Why that hint did not come from Einstein's friend Grossmann, who should have known the contracted Bianchi identities

$$(R_a^b - \frac{1}{2}Rg_a^b)_{;b} = 0 (9)$$

which follow from the Bianchi identities, discovered by Bianchi^[28] in 1902.

In 1899 a German translation (first edition) of Bianchi's 'Differential geometrie' appears, of course without the Bianchi identities. In his second 1910 German edition, Bianchi has omitted differential geometry for curved space with n > 2. Obviously such a subject seemed too devoid of any application to be included in a textbook. Only in a footnote, Bianchi has added the symmetry properties of the Riemann tensor, which are trivial for n = 2. His Bianchi-identities seemed too uninteresting to be included even in that footnote.

For the years 1850-1915, I have scanned all German journals, the Sommerville bibliography^[29], and all German and non-German text books reviewed in Acta Mathematica, and I did not find any mentioning of Bianchi's (contracted) identities. 1869 was the start of the excellent reviewing journal "Jahrbuch über Fortschritte der Mathematik" reviewing most German and non-German mathematical works. Bianchi's discovery was reviewed in Vol 33(1902) but completely un-conspicuous among the enormous amount which was published in mathematics already at that times. Until 1915 not a single mentioning of the identities. They appear for the first time in 1918 in Schouten's textbook^[30].

Also Hilbert had no chance to know (9). So the assumption that he had used it for his derivation of the explicit field equations are untenable. However, *after* he had derived the field equations, because of Theorem III of his page proofs, he new the contracted Bianchi identities implicitly. But definitely, they were not among his explicitly known theorems.

4.2 Has Einstein found his equations independently from Hilbert?

In Nov 11, 1915, Einstein still had the equations in the form (2), lacking the trace term. He was dissatisfied because, using the gauge (4), which is always possible, he could conclude

$$T = \text{const.}$$
 (10)

i.e. $T \equiv 0$ by integrating to a vacuum point.

In his Nov 25 paper, Einstein does not give a method how he has arrived at the additional trace term in (7). He merely shows that with the trace term, conclusion (10) can no longer be drawn. He does not show (3), which amounts to proving the contracted Bianchi identities (9). Thus in Nov 25, Einstein did not know his theory was already complete, though he^[3] writes on p. 847: "Thus finally, general relativity is a closed logical edifice. The relativity postulate in its most general formulation, according to which coordinates are irrelevant parameters, leads with absolute necessity to a uniquely specified theory of gravitation". Obviously, he relied on Hilbert's Nov 16 postcard for this (partially incorrect) conviction. And this again excludes the possibility Hilbert has sent with his Nov 16, 1915, postcard only undetermined coefficients in front of the Ricci tensor and in the trace term.

5 A short introduction to the theory of invariants

Hilbert's derivation of the Einstein equations are essentially based on invariant theoretical arguments.

5.1 Algebraic invariants

The invention of analytical geometry, i.e. the application of algebra and infinitesimal calculus to geometry, was a major progress in solving geometrical problems. However, it had a great disadvantage, because it required the introduction of a coordinate system which is arbitrary and not or not uniquely related to the geometrical objects under consideration. This was the birth of the theory of invariants. We mention the names of Sylvester and Cayley. It was mainly an algebraic invariant theory and only a binary one, applicable to the Euclidean n = 2 dimensional plane. The best known algebraic invariant is the sum of the squares of the coordinate difference of two points being the geometrical objects, an invariant which has the geometrical interpretation as the square of the distance between these points. Algebraic invariant theory culminated with the textbooks 1872 by Clebsch and 1885 by Gordan^[31]. Most mathematicians at that time and in particular Hilbert contributed to that field. Hilbert has become famous for the first time by his finiteness theorem of the system of algebraic invariants (solving Gordan's problem for n > 2).

5.2 Gauss's measure of intrinsic curvature

The first milestone in differential invariants was 1828 Gauss: Disquisitiones generales circa superficies curvas^[32], where he considered a general n = 2 dimensional curved surface embedded in 3-dimensional Euclidean space. (Disregarding global topology this is the most general 2-dimensional Riemannian space.) He introduced the following differential invariant:

$$k = \frac{1}{4} (EG - F^2)^{-2} \left[E \left(\frac{\partial E}{\partial q} \frac{\partial G}{\partial q} - 2 \frac{\partial F}{\partial p} \frac{\partial G}{\partial q} + \left(\frac{\partial G}{\partial p} \right)^2 \right) + F \left(\frac{\partial E}{\partial p} \frac{\partial G}{\partial q} - \frac{\partial E}{\partial q} \frac{\partial G}{\partial p} - 2 \frac{\partial E}{\partial q} \frac{\partial F}{\partial q} + 4 \frac{\partial F}{\partial p} \frac{\partial F}{\partial q} - 2 \frac{\partial F}{\partial p} \frac{\partial G}{\partial p} \right) + G \left(\frac{\partial E}{\partial p} \frac{\partial G}{\partial q} - 2 \frac{\partial E}{\partial p} \frac{\partial F}{\partial q} + \left(\frac{\partial E}{\partial q} \right)^2 \right) - 2(EG - F^2) \left(\frac{\partial^2 E}{\partial q^2} - 2 \frac{\partial^2 F}{\partial p \partial q} + \frac{\partial^2 G}{\partial p^2} \right) \right]$$
(11)

where $E = g_{11}$, $F = g_{12}$, $G = g_{22}$ and $p = x^1$, $q = x^2$ are arbitrary curvilinear coordinates on the surface. The invariant has the geometrical significance of k what is now called the 'Gaussian measure of curvature'. Taking a finite piece of the surface, he collected all unit normal vectors (orthogonal to the surface) and fixed them at the origin of a unit sphere. The area of their end points he called total curvature (*curvatura integra*) of the surface. k (at a point P) is the quotient of that area by the area of the surface itself, taking an infinitesimal neighbourhood of P. k is also the product of the minimal and maximum exterior curvature of geodesic lines emanating from P. He showed that k is an invariant against bending (think of a thin material surface resisting to stretch and tear) and of course against rotation and translation. Since every n = 2 Riemannian space can be (locally) embedded in R^3 , 'bending' (which is not necessarily a continuous process, and is also referred to as 'developing one surface unto another' e.g. developing a cylinder unto a plane) is synonymous with 'isometry'. Thus (11) is a 'bending invariant', and k was called 'internal curvature of the surface'. Trivially, since no assumptions about the choice of the coordinates p, qhave been made, (11) is also an invariant against arbitrary coordinate transformations. Gauss called this invariance properties *theorema egregium* (remarkable theorem.)

5.3 Riemann and Christoffel: the *n*-dimensional case

In 1854 in his $Habilitationschrift^{[33]}$ (where because of the audience he could use words only) Riemann introduced qualitatively the concept of an n dimensional Riemannian space, by considering the quadratic form

$$ds^2 = \sum_{a,b=1}^{n} g_{ab} dx^a dx^b \tag{12}$$

describing the metric. He generalizes Gauss's work by constructing all geodesic (2-dimensional) surfaces emanating (with all orientations) from a point P, considering their k given by (11).

The hard formula work was done in 1869 by E.B. Christoffel^[34] where he introduced his 3-indexsymbols, the 4-index-symbols (now called Riemann tensor), the covariant derivative, permitting to construct invariants and covariants, i.e. tensors, of arbitrary order. (Only later and posthumously, it was known that Riemann had done a substantial part of this work before^[35].) Gauss's k is regained as the (only independent) component of the Riemann tensor:

$$k = g^{-1} R_{1212} \tag{13}$$

5.4 Gaussian invariants

A Gaussian invariant of order G of n variables x^a is a function $J(x^a, g_{ab}, \dots, g_{abk_1 \dots k_G})$ depending on arbitrary functions $g_{ab}(x^k)$ and their partial derivatives (denoted by additional indices) up to the order G, with the condition that the value of J remains unchanged if we introduce new variables x'^{μ} (coordinate transformation) whereby the $g'_{\mu\nu}$ are found by the well known formulae for transforming a metrical tensor, and their derivatives are now calculated with respect to the new coordinates. Symbolically we could write

$$J(x^a, \mathcal{G}) = J(x^{\prime a}, \mathcal{G}') \tag{14}$$

where \mathcal{G} summarizes all metrical variables (including their derivatives). On both sides J is the same mathematical function ('form invariance') and the primed und unprimed arguments on both sides are connected by the coordinate transformation, and are taken at the same objective geometrical point, having two different coordinates (x^a and x'^a).

The Ricci scalar is a Gaussian invariant of order G = 2. In this case the function J is the (forminvariant, i.e. coordinate independent) prescription how the Ricci scalar is calculated. There are no Gaussian invariants of lower (G < 2) order.

As an exercise we will prove, that an invariant J cannot depend explicitly on the coordinates x^a . Taking a fixed invariant J, (14) is an infinite set of numerical equalities, where the set runs over all possible choices for the functions g_{ab} , all values of the coordinates x^a and all possible coordinate transformations. Arrange the elements of this set in pairs, where two members of the pair differ by the choice of the coordinate transformation only, namely by an additional translation $(x''^a = x'^a + c)$ with a fixed chosen c and a. Both members have the same left hand side of (14). By writing down the transformation law for a metric tensor for ' and for '' (and for its derivatives) one sees that for the collection of numerical values (denoted by \mathcal{G}) there holds $\mathcal{G}' = \mathcal{G}''$. Thus the right hand side of the members differ only in the variable x'^a by c. Since c is arbitrary, J cannot depend on x^a .

5.5 Beltrami invariants

To a quantity constructed from the components of the metrical tensor (and its partial derivatives) alone (also called a 'concomitant' of the metric) we give the specifier 'Gaussian'.

To subsume Gaussian tensors (also called Gaussian 'covariants') to the concept of invariants, we have to consider Beltrami invariants:

A Beltrami invariant (in the literature also denoted by the strange name 'differential parameter of the first kind') of orders (G, B) is a function J which additionally depends on a scalar function $\varphi(x^a)$ and its partial derivatives $\varphi_a, \dots \varphi_{k_1 \dots k_B}$ up to the order B.

(Note that a scalar function is also transformed, i.e. is replaced by a new mathematical function φ' :

$$\varphi(x^a) = \varphi'(x'^a) \tag{15}$$

though it is usual in physics to omit the prime on φ distinguishing different functions.) Beltrami (based on Lamé) has found the square of the gradient $(g^{ab}\varphi_a\varphi_b)$ and the Laplacian $(g^{ab}\varphi_{a;b})$, what is now called the first (G = 0, B = 1) and second (G = 1, B = 2) Beltrami invariant.

The concept is enlarged to include S scalar functions φ^s . A Gaussian *covariant* (e.g. the metrical, Riemann or Ricci tensor) can be viewed as a Beltrami invariant which is contracted with the gradients (B=1) of S (S = order of the tensor) scalar fields. Since we have to deal with symmetrical second rank tensors T_{ab} only, we can restrict ourselves to S = B = 1, since the invariant $T^{ab}\varphi_a\varphi_b$ for arbitrary φ_a determines the tensor T_{ab} uniquely.

We say that there are N = N(n, G, B, S) Beltrami invariants J_1, \dots, J_N if every Beltrami invariant J of this or lower order can be expressed as a function \mathcal{J} of them:

$$J = \mathcal{J}(J_1, \cdots, J_N) \tag{16}$$

and these N invariants are independent, i.e. no one can be expressed (in the sense of (16)) by the remaining N-1 ones.

The S scalar functions (e.g. take $J = J(\varphi) = \varphi$) are themselves invariants (though not differential invariants). It is usual to discard them as trivial. When the number N of a complete system of independent invariants J_1, \dots, J_N is calculated, they are not counted. Even more radically, an invariant is called a Beltrami-invariant only if it does not depend on the φ^s (but on their derivatives $\varphi^s_{a_1 \dots a_B}$ only).

This terminology has the consequence that a Beltrami invariant B = 0 is a Gaussian invariant.

As an exercise, we prove this is a good practice. For clarity we distinguish free Beltrami invariants ('f-invariants') from general Beltrami invariants ('g-invariants') which may depend on the φ^s . We will prove the following lemma: When J_1, \dots, J_N is a complete system of independent f-invariants, then an arbitrary g-invariant is again given by a function \mathcal{J} in (16) with the φ^s as additional arguments.

For simplicity we take the typical case the g-invariant J fulfills the equations:

$$J(\mathcal{G},\varphi,\varphi_a) = J(\mathcal{G}',\varphi',\varphi'_a) \tag{17}$$

i.e. S = 1, B = 1 As in our previous exercise, (17) is an infinite set of numerical equalities. Now we show the same set also corresponds to the equations of the following f-invariant (choosing a fixed c):

$$J(\mathcal{G},\varphi_a;c) := J(\mathcal{G},c,\varphi_a) = J(\mathcal{G}',c,\varphi'_a)$$
⁽¹⁸⁾

Consider one element of the set, specified by coordinates x^a and a function φ (with corresponding values x'^a and $\varphi'(x'^a) = \varphi(x^a) =: C$ on the right hand side of (18)). When c = C (18) is fulfilled. Otherwise pick the element with function $\Phi = \varphi + c - D$ out of the set (same x^a). Φ and φ have the same derivatives. So the picked element says (18) is again fulfilled.

Since c is arbitrary, \overline{J} are an infinite set of f-invariants (except when \overline{J} does not depend on c, i.e. J was already an f-invariant). Since sum, difference, constant multiples of invariants are again invariants, derivatives (to arbitrary order) of \overline{J} with respect to c are again f-invariants. Assuming power series for J in its middle argument, we have proved our lemma.

6 How Hilbert has found the Einstein equations

6.1 Hilbert's invariant theoretical prerequisites to find the Einstein equations

Hilbert derives the Einstein equation by a variational principle with a Lagrangian density

$$L = R + L_{\rm em} \tag{19}$$

(Hilbert writes H = K + L, L_{em} is an electromagnetic Lagrangian density, later specialized to that one of Mie's theory.) Thus Hilbert has to calculate the variational derivative

$$\frac{\delta\sqrt{g}R}{\delta g^{ab}} = \frac{\partial\sqrt{g}R}{\partial g^{ab}} - \sum_{k=1}^{n} \frac{\partial}{\partial x^k} \frac{\partial\sqrt{g}R}{\partial g^{ab}_k} + \sum_{k,l=1}^{n} \frac{\partial^2}{\partial x^k \partial x^l} \frac{\partial\sqrt{g}R}{\partial g^{ab}_{kl}} =: \sqrt{g}G_{ab}$$
(20)

where for reasons of convenience we have introduced the quantities G_{ab} for the result of this calculation. To derive the Einstein equations, he had to show:

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}.\tag{21}$$

In Theorem III of his page proofs he shows that G_{ab} is a second rank tensor. Obviously it is a symmetric one.

R contains the second derivatives $g_{kl}^{\mu\nu}$ only linearly and the first derivatives $g_k^{\mu\nu}$ only bilinearly with coefficients containing the zero-th derivatives only. Thus inspection into the variational procedure (20) easily leads to the conclusion G_{ab} can contain the second derivatives only linearly. (Note that g^{ab} are rational functions of the g_{ab} and vice versa, with g or 1/g in the denominator.)

In invariant theoretical language, Hilbert's assumption (1) says that up to order G = 2, B = 1, S = 1 there are only 3 independent invariants. As these we can take:

which are independent. So every invariant J of that type is a function of them: $J = \mathcal{J}(J_1, J_2, J_3)$. The invariants (22) are linear in the second derivatives of the metric. If J should also have this property and be bilinear in the φ_a , it follows (assuming power series and because the J_i can assume independently any values) that

$$G^{ab}\varphi_a\varphi_b = J = [\alpha R^{ab} + \beta R g^{ab} + \lambda g^{ab}]\varphi_a\varphi_b.$$
⁽²³⁾

Since the φ_a are arbitrary, and since both G^{ab} and [] in (23) are symmetric we have:

$$G_{ab} = \alpha R_{ab} + \beta R g_{ab} + \lambda g_{ab} \tag{24}$$

Note that the function \mathcal{J} (for any chosen J) in (16) does not depend on any of the arguments of an invariant, i.e. not on $g^{ab}, g_c^{ab} \cdots$. This fact, trivial for Hilbert, was perhaps the essential stumbling-block, why Hilbert's derivation of the Einstein equations was not understood thus far since 90 years.

For a fixed n (dimension of space) J is a fixed invariant. Thus α, β, λ are numerical constants, possibly depending on n only.

Thus Hilbert's method how he has found the Einstein equations 'easily without calculation' becomes obvious: Take a metric g_{ab} as simple as possible, but non-trivial enough so that the constants α, β, λ are determined uniquely while (20) is evaluated. Otherwise the metric is completely arbitrary. It had not to be physically meaningful in any way. That is done in the following subsection.

6.2 How Hilbert has found the *explicit form* of the Einstein equations

From the Göttingen $\operatorname{archive}^{[12]}$ one can obtain a Xerox copy of a microfilm of Hilbert's private folder "*Zur Elektrodynamik*". (Hilbert erroneously^[36] believed to have shown the electromagnetic phenomena to be a consequence of gravitation.) In this folder Hilbert's derivation can be found on the page with Archive-number 32. Here he collects "*Formeln*" he has derived for the Ricci-tensor and the Ricci-scalar in the case of a general diagonal metric:

$$R_{ii} = \frac{1}{2} \sum_{k,(k\neq i)} \left\{ g^{kk} (g_{kkii} + g_{iikk}) - \frac{1}{2} g^{kk} \left(2g^{ii} g_{iii} g_{kki} + g^{kk} g_{kki} g_{kki} + 2g^{kk} g_{iik} g_{kkk} + g^{ii} g_{iik} g_{iik} \right) + \frac{1}{2} g^{kk} \sum_{\rho} g^{\rho\rho} g_{ii\rho} g_{kk\rho} \right\}.$$
(25)

Then he mentions a "Spezialfall' (special case)

 $g_{11} = g_{22} = g_{33} = 1, \quad g_{44} = \gamma, \quad g^{44} = 1/\gamma$ (26)

and for this special case he gives the Ricci scalar

$$R = \sum_{i < 4} (\gamma^{-1} \gamma_{ii} - \frac{1}{2} \gamma^{-2} \gamma_i^2).$$
(27)

We simplify Hilbert's special case even a bit more by assuming

$$\gamma = \gamma(x) \text{ with } x \equiv x^1.$$
 (28)

Introducing $\kappa = g^{44} = 1/\gamma$ (or starting afresh from the definitions in textbooks; Hilbert's formulae differ by a minus sign from the convention in Landau-Lifshitz^[37]) one immediately finds (' = derivative with respect to x):

$$\sqrt{g} = \kappa^{-\frac{1}{2}} \tag{29}$$

$$R = -\kappa^{-1}\kappa'' + \frac{3}{2}\kappa^{-2}\kappa'^2 \tag{30}$$

$$R_{44} = -\frac{1}{2}\kappa^{-2}\kappa'' + \frac{3}{4}\kappa^{-3}\kappa'^2 \tag{31}$$

Now we do the variational derivative in (20) for $\mu = \nu = 4$, which in this case yields zero:

$$\frac{\partial\sqrt{g}R}{\partial\kappa} - \frac{\partial}{\partial x}\frac{\partial\sqrt{g}R}{\partial\kappa'} + \frac{\partial^2}{\partial x^2}\frac{\partial\sqrt{g}R}{\partial\kappa''} = 0 \stackrel{!}{=}$$

$$\stackrel{!}{=} \kappa^{-\frac{1}{2}} \left[\boldsymbol{\alpha}(-\frac{1}{2}\kappa^{-2}\kappa'' + \frac{3}{4}\kappa^{-3}\kappa'^2) + \boldsymbol{\beta}\kappa^{-1}(-\kappa^{-1}\kappa'' + \frac{3}{2}\kappa^{-2}\kappa'^2) + \boldsymbol{\lambda}\kappa^{-1} \right]$$
(32)

 α, β, γ are independent of the function κ , so we can equate coefficients of κ'' and κ' which leads to $\lambda = 0$ and

$$\alpha + 2\beta = 0 \tag{33}$$

(Note that we cannot check the other components $\mu\nu$ since our simple formulae (27) are not valid for variations of the other components of $g^{\mu\nu}$.)

By this simple method, Hilbert has derived the Einstein equations and has communicated them to Einstein on his 1915 Nov 16 postcard. (33) is equivalent to what the contracted Bianchi identities had given, i.e. the correct trace term. For Einstein the absolute values of α and β are irrelevant, since Einstein determined the constant of gravitation κ by comparison with Newton's theory. Since Hilbert, in contrast to Einstein, also gives an explicit Lagrangian for the sources of the gravitational field, he needs the absolute values $\alpha = 1$ and $\beta = -\frac{1}{2}$. He might have found them by calculating another special case. We have no hint to that. Therefore, in the next subsection, we propose another even more elegant route Hilbert might have guessed these correct values.

The derivation, just given, cannot be found in Hilbert's private folder. Hilbert sometimes has calculated on letter sheets he has obtained from Swiss hotels (bearing a logo of the hotel in the letter head). Paper was extremely expensive. Hilbert had in his garden a big blackboard under a roof where he made his intermediate calculations^[38]. Thus on his sheet 32 we only find important formulae such as (25), and the indication of the 'Special case' he had used, but not the trivial calculation itself.

This derivation by Hilbert is not only historically the first one, it is also a nice exercise for an elementary class in General Relativity, presupposing that the field equations have the form (24), and fitting the constant of gravitation by Newton's limit.

6.3 'easily without calculation'

In this subsection we give a route how Hilbert immediately might have arrived at (39) using his tremendous power of mathematical intuition. ($\lambda = 0$ was obvious to him, by inspection into the procedure (20) of variational derivation.)

In an attempt to calculate the left most side of (20) he writes down

$$\delta\sqrt{g} = -\frac{1}{2}\sqrt{g}g_{ab}\,\delta g^{ab}.\tag{34}$$

The theory of determinants, which are the most important invariants, was at the focus of interest at that time. Riemannian spaces for n > 2 have been studied almost exclusively for the isotropic case

$$R_{ab} = \frac{R}{n} g_{ab},\tag{35}$$

for which F.Schur's theorem^[39] that R is spatially constant, was famous. For variation of the metric

$$\delta g^{ab} = g^{ab} \delta \chi \tag{36}$$

with a spatially constant $\delta \chi$, we immediately find

$$\delta R = R \, \delta \chi \tag{37}$$

since R scales with g^{ab} . So from (20) (24), multiplied by (36), we find

$$1 = \alpha + (\beta + \frac{1}{2})n. \tag{38}$$

Together with (33) we find

$$\alpha = 1, \quad \beta = -\frac{1}{2}. \tag{39}$$

Since in the variational procedure (20) and in its result (24) the dimension n of space enters only as the boundary of a formal sum, perhaps Hilbert, using his intuition, knew that α and β cannot depend on n. So he has arrived immediately from (38) to (39) even without (20), indeed, 'easily without calculation'.

For the reader uneasy with (37) we proceed more formally: Start from the middle expression in (20), and (24) with $\lambda = 0$. Multiply by (36) and integrate over the whole space of (35), taken as the *n*-dimensional sphere Sⁿ. Reverse the steps of partial integration, which had led to the middle expression of (20). No boundary terms appear, since we integrate over a closed space, even for a constant $\delta \chi$. Instead, the complete differential $\delta \sqrt{gR}$ appears. Consider the range of metrics χg^{ab} , where g^{ab} is fixed and given by (35), and χ is spatially constant. Thus the complete differential δR is given by (37). Push the constants R and $\delta \chi$ in front of the integrals to arrive at (38).

7 Discussion of Hilbert's derivation

7.1 Hilbert challenged by Zorawski and Haskins

Sophus Lie (1842-1899) made a tremendous contribution to the theory of invariants by recognizing that a (transformation) group (its 1-component) is uniquely determined by its infinitesimal transformations. Thus an invariant is already identified when it is invariant with respect to infinitesimal transformations. This opened the possibility to determine the number N of (independent) invariants by counting the number of equations and the number of freedoms of the invariants, and using lemmas guaranteeing the functional independence of the invariants thus found. This method only gives the *number* of (independent) invariants, not their (global) explicit form, except in cases where this is already known otherwise. These numbers have been found in 1891 by Zorawski^[14] for n = 2 and for general n in 1904 by Haskins^[13].

Table 1 summarizes some of their results.

As an illustration we discuss some of the boxes. The scalar field φ , being itself an invariant and belonging to the box (G = 0, B = 0) is a trivial invariant and is not counted as explained in subsection (5.5).

The first Beltrami invariant $g^{ab}\varphi_a\varphi_b$ is the one invariant in the box (G=0, B=1).

In the box (G = 2, B = 0) we find the Ricci scalar R, which only for n = 2 is the only invariant of that type. Already for n = 3 we have 3 independent Gaussian invariants. They have been given explicitly in 1873 by Souvaroff^[40]:

$$S_{1} = R, \quad S_{2} = g^{-1} R_{kl} \varepsilon^{la\alpha} \varepsilon^{kb\beta} R_{a\alpha b\beta},$$

$$S_{3} = g^{-2} R_{klmn} \varepsilon^{ka\alpha} \varepsilon^{lb\beta} \varepsilon^{mc\gamma} \varepsilon^{nd\delta} R_{a\alpha c\gamma} R_{b\beta d\delta}$$

$$\tag{40}$$

Except the Ricci scalar they are non-linear in the second derivatives.

For n = 4 we have 14 invariants. They are given explicitly 1956 by Géhéniau and Debever^[41]. In Misner, Thorne, Wheeler^[42] it is stated that all of them except R are non-linear and do not have Newton's gravitation as a limit.

N	G = 0	G = 1	G=2	G=3	$G \ge 4$
B = 0	0	0	$\frac{1}{12}(n-2)(n-1)n(n+3)$	$n \frac{(n+2)!}{4!(n-2)!}$	$n\frac{G-1}{2}\frac{(n+G-1)!}{(n-2)!(G+1)!}$
			N=1 for $n=2$	$N{=}1$ for $n{=}2$	
B = 1	1	0	n-1	0	0
			N=0 for $n=2$	$N{=}1$ for $n{=}2$	
B=2	0	2n - 1	(n-1)(n-2)/2	0	0
		$N{=}3$ for $n{=}2$			
B=3	0	0	$\frac{(n+2)!}{(n-1)!3!}$	0	0

Table 1: Number N of independent differential invariants in n-dimensional space $(n \ge 2)$. G = 2means the invariant involves up to *second* partial derivatives of the metrical tensor with respect to the coordinates. B = 0 are the Gaussian invariants, which do not involve scalar fields. B = 1are Beltrami-invariants involving the *first* derivatives φ_a of a single scalar field φ . The whole table counts only Beltrami invariants involving (derivatives of) a *single* scalar field (S = 1). A box (e.g. G = 3, B = 2) of the table does not count invariants which are dependent (can be constructed via (16)) by invariants in lower boxes ($G \le 3, B \le 2$). The case n = 2 is exceptional, so sometimes its N is given separately.

The Ricci-tensor, i.e. the Beltrami invariant $R^{ab}\varphi_a\varphi_b$ is absent in the box (G = 2, B = 1) for n = 2. Because of $R_{ab} = \frac{1}{2}Rg_{ab}$, it can be constructed by lower invariants.

7.2 Was Hilbert's theory not fully covariant?

Some historians claim Hilbert, while writing his page proofs or sending his Nov 16, 1915, postcard, cannot possibly have known the correct field equations of General Relativity, since in his Axiom III of the page proofs he defines special coordinates, which he calls space-time coordinates, thus he had not yet obtained full covariance. These historians overlook, what Hilbert and Einstein (in his Nov 4 paper) have clarified almost 90 years earlier: A generally covariant theory must be supplemented by non-covariant gauge-conditions, i.e. by specializing the coordinates to some degree. A generally covariant theory means that I can choose the coordinates completely arbitrary, also in the future. No physical theory can predict my choice of the coordinates in the future ('lack of causality'). Therefore, I must tell my choice for the future, and this is done by imposing non-covariant so called gauge-condition, additionally to the field equations. Such gauge conditions should not restrict the generality of the gravitational field (a restriction which is the duty and the right of the field equations only), i.e. the gauge-condition must be a *permitted* gauge condition. To prove that a gauge-condition is permitted, one has to show that in an arbitrary gravitational field expressed with arbitrary coordinates, one can always transform to new coordinates fulfilling the gauge conditions.

Hilbert's space-time coordinates are a gauge condition, but not a permitted one. He recognized that. Therefore, he suppressed it in the published version of the page proofs. This, however, by no means affects the general covariance of his field equations.

7.3 Are the theories of Hilbert and Einstein different?

In his second Nov 13 postcard to Einstein, Hilbert's claims his theory to be completely different from Einstein's. However, this remark refers to his treatment of the energy concept of the gravitational field only. Concerning the field equations, Einstein assumes a general (i.e. unspecified) energy-momentum-tensor T_{ab} while Hilbert assumes the sources of the gravitational field to be expressible by a Lagrangian density L, see (6). While deriving the gravitational field equations, Hilbert assumes that L depend only on the $g^{\mu\nu}$ (not its derivatives) and on some variables q_s and its first derivatives. Even in contemporary physics, there is no viable (i.e. non-speculative) model of matter contradicting these assumptions. Therefore, Hilbert's approach (as is Einstein's) is completely general with respect to the sources of the gravitational field. It is true, Hilbert calls the q_s 'the four electromagnetic potentials' and later he specializes L to that one of Mie's theory, into which nowadays nobody believes. However, these verbal denotations and later specializations do not affect Hilbert's derivation of the field equations, being general with respect to the sources. Hilbert's approach was much more far-seeing than Einstein's, since Hilbert postulates a unified Lagrangian formulation of the whole of physics (which at that time was gravitation and electromagnetism only), so he must be considered the father of the concept of a unified field theory including gravitation, to which Einstein turns only much later.

7.4 Has Hilbert found General Relativity before Einstein?

Definitely Einstein had the idea to interpret gravitation as geometry of 4-dimensional non-Euclidean space, Einstein had the almost correct (2) field equations of general relativity, and he had pursued the theory consistently during eight years. The question could only be if Hilbert had found the finally correct field equations (7) before Einstein. We have given stringent arguments for this question to be answered affirmative. However, we have also argued it was by Hilbert's good luck that he has achieved this. His derivation was based on the fact stated in (1), which is incorrect as it stands, or at least, when modified by the requirement of linearity in the second derivatives, was not proved in 1915. Hilbert has displayed an astonishing mathematical intuition by formulating his famous 23 problems. All of them have tuned out to be non-trivial, and most of them solvable. This was a remarkable achievement. So also in this case he was guided by his intuition, which in the end was not misleading.

From the point of view of physics, Hilbert did not bother about the question if his field equations have the correct Newtonian limit. So again it was good luck, that his choice of R as the Lagrangian turned out to be the physically correct one.

It seems that this was also Hilbert's own judgment: It is reported^[38] Hilbert had joked: "Every boy in the streets of Göttingen understands more about four-dimensional geometry than Einstein. Yet, in spite of that, Einstein did the work and not the mathematicians."

Laws of nature cannot be deduced or derived. They can only be guessed, hoping that future experiments would verify or at least not falsify them. This is because nature did not anticipate our wishes.

Hilbert was the first to have guessed the correct laws of gravitation, and had communicated them on his Nov 16 postcard, which was lost or was intentionally abolished.

8 Mutilation of Hilbert's Page Proofs

Since recently, very accurate photos of Hilbert's page proofs are available on the internet^[43]. They can also be ordered as a copy from the Göttingen archive^[12].

About one third of a doubly printed sheet (top of page 7 and 8) are cut-out along a wavy line, so it was not done by scissors but by a razor blade or a knife. Since on page 7 the cut goes irregularly middle of a printed line, whereas on page 8 the cut goes, though wavy, but exactly between two lines of text, it is clear the cutter intended page 8. As judged by context and by comparing it with the published version, the cut-off contained the definition of the gravitational part of Hilbert's Lagrangian density, which he denoted by K. In the remaining part of the page proofs it can no longer be seen, that Hilbert has chosen K to be the Ricci scalar R. The page proofs have been investigated thoroughly by the Göttingen historian Daniela Wuensch, who has published the results in her book^[21] "Zwei wirkliche Kerle" In the following we give a reformulation of her proof of circumstantial evidence that Hilbert's page proofs have been mutilated in recent years. The logic of the proof is as follows: Several curious facts can be seen today on the page proofs which cannot be explained in a rightful way, but which find a plausible explanation by the motivation of a historian intending to withdraw Hilbert's priority and the ensuing need to cover-up.

8.1 Reasons why the cut-off was done between 1994-1998

1) The cut-off was mentioned for the first time in T. Sauer^[26], an article which has appeared in 1999.

2) In 1998 the page proofs have been filmed by the Göttingen archive, and on the film the cut-off is already visible. Thus the cut-off was done before 1998.

3) The main statement in the *Science* 1997 article^[5] was that in Hilbert's page proofs the explicit field equations cannot be found. It is improbable the editors and referees of *Science* had not required a copy of the page proofs, and they had remarked the cut-off and had insisted on mentioning it. However, it is possible that personal relationships had exceptionally made a short-cut of the usual refereeing procedures. An investigation at *Science* should be initiated. Perhaps a referee or the editorial board is still in possession of an intact copy of the page proofs.

4) In this article a very minute comparison is made between the page $\text{proofs}^{[2]}$, the published version^[6] and the republished version^[15] of 1924. In particular on p. 1272 middle column line 5, the authors of the article^[5] write:

"In the proofs of his first communication, Hilbert's world function includes a gravitational term $\sqrt{g}K$...".

However, they could not see that, because on the incomplete page proofs it cannot be found, and as judged by the published version, it was stated on the cut-off.

5) The authors would have mentioned the cut off. For historians it would be an opportunity to speculate and to publish about such a peculiarity in a historical document. On the contrary they let the priority of its discovery to others.

6) Not mentioning the cut-off in an article, stating the whole historical document does not contain the field equations, would be equivalent to a grossly unethical scientific behavior which would not be committed without a strong motivation.

Thus CRS have been in possession of an intact copy of the page proofs. Since they have discovered the page proofs shortly after 1994, the cut-off was not yet done before 1994.

8.2 Reasons why Hilbert did not do the cut-off himself

Besides the evidence just given, it is preposterous to assume Hilbert himself could have done it. Out of over 62 publications, just this once he has kept the page proofs, in all other cases the reprints of the published versions only. He had namely omitted something in the final publication because it seemed insecure to him and he intended to improve it later.

Fetching razor blade and glue, making the cut-off and inserting it somewhere else would cost more time than copying some simple formulae.

In 1918 Hilbert sends the page proofs^[2] to Felix Klein, who was interested to see exactly what Hilbert has omitted in the published version^[6]. In the accompanying letter^[44], dated Mar 7, 1918, Hilbert asks Klein expressively to be careful with the page proofs and to send them back, because he had no other records. But the forger has overlooked that on the backside of the doubly printed cut-off was exactly what Hilbert wanted to show Klein.

8.3 Ripping apart of a sheet and hand-foldings

Besides the cut-off, Hilbert's page proofs have suffered additional disfigurements. Sheet one of the page proofs have been torn into two pieces. There is no reasonable explanation for this fact, when everything had gone the right way. However there is a plausible explanation for a forger by his motivation for cover-up.

Originally the page proofs consisted of three big 4-page sheets and of one final small sheet. The first big sheet (frontside-backside, i.e. doubly printed) contained pages 1-2 and beneath pages 7-8, so that after sewing, all pages come into the correct order. (On the small sheet was page 13, backside empty.) The big sheets have been machine-folded vertically in the middle by the printery. Up to now, the third sheet has, beside the machine folding, no additional (i.e. hand-) folding. Thus we can conclude the printery has sent (Dec 6, 1915) the page proofs in a big envelope (approx. 24 cm x 17 cm).

After the cut-out, as the first additional action, sheet one was ripped-apart along the middle vertical machine-folding (i.e. page 1-2 was separated from page 7-8, the latter with the cut-off).

Second, page 1-2, page 7-8 and sheet two have been folded by hand to the format 19 cm x 15 cm.

The temporal sequence of these manipulations is undoubted: First the cut-off (almost one third of the page 7-8), then ripping apart of sheet one (ripping off the remaining part of page 7-8 from page 1-2), then folding to the smaller format (19 cm x 15 cm). The cut-off was made by a razor blade (as can be seen by the wavy cutting curve), the separation was made by simple tearing off by hand along the machine-folding. The cut-off has two razor blade edges (one vertical, one horizontal), but no (vertical) tearing edge. Thus, first cut-off, then separation of sheet one.

Only then came the foldings, because page 1-2 and page 7-8 (being only 2/3 of a normal page) are folded differently, namely in their respective middles.

At Mar 7, 1918 Hilbert writes a letter to Felix Klein^[44] in the format 19 cm x 15 cm:

"Hereby, I send you the first proofs * (3 sheets) [= the page proofs^[2] discussed here] of my first communication, (* Please, kindly to return them to me, because I have no other records) in which I just have elaborated also what are now Runge's ideas, in particular Theorem 1, page 6, where I have proved the divergence property of energy. But later [i.e. In the published version^[6]] I have suppressed the whole thing, because it did not seem mature to me. I would be very pleased if now a progress could be achieved."

Theorem 1 begins on bottom of page 6 and is proved on top of page 7, i.e. at that position of the page, which is now lacking. Hilbert does not send a reprint but the page proofs, because in the final publication he had omitted ("suppressed") something. Even more absurd it would be to assume Hilbert has made the cut-off before sending the letter to Klein, so exactly that would be missing he intended to show Klein. At least he would have mentioned the cut-off with regret, while mentioning the 3 sheets, he is sending.

According to the temporal sequence established above, Hilbert did not fold the page proofs to the format of the letter (19 cm x 15 cm). Hilbert has sent the page proofs unfolded in a big envelope (24 cm x 17 cm, perhaps in the same he had received from the printery). The accompanying letter

in the smaller format (19 cm x 15 cm) he had simply enclosed.

Is it possible Hilbert had sent the page proofs to someone else (after Klein) and in order to save postage, he had folded the page proofs? To everyone else, Hilbert would have sent a reprint of the final published version, except the addressee, as Klein, is exactly interested what Hilbert had suppressed in the final version, namely what is written on the cut-off page 7. So the same arguments as for Klein are valid also.

Now, we already have three mysteries: Who (Hilbert, his wife Käthe, a maid, an assistant) has performed (and why ?) the cut-off, has ripped apart sheet one, and has folded the pieces to the format 19 cm x 15 cm? The cut-off, the separation of sheet one and the hand-foldings are a fact, which cannot find a reasonable explanation when everything has gone the right way. However, there is a plausible explanation for a forger.

The forger, after having done the cut-off has pondered how he could further manipulate the page proofs in order to prove the cut-off were done in historical times, namely before Mar 7, 1918 (Hilbert's letter to $Klein^{[44]}$). The forger must have been in possession of relevant historical details. He must have known Hilbert has sent a letter to Klein in the format 19 cm x 15 cm, and he must have known the contents of that letter.

Obviously, the forger did not pay attention to the argument above (about what Hilbert wanted to show Klein). But he has taken care of for a proof of the temporal sequence: cutting-off, ripping-apart, folding. It has seemed plausible to the forger, Hilbert has folded the page proofs to the smaller format of the accompanying letter, perhaps to save postage. Thus, seemingly, the forger could prove the cut-off was done before Mar 7, 1918, because the foldings presuppose the cut-off.

Notice, the forger had to rip-apart sheet one, because only now, he could fold the mutilated page 7-8 (with the cut-off) in the middle of the remaining 2/3 of the page, whereby he could make sure, the foldings have taken place before the sending of the letter (Mar 7, 1918).

8.4 The Roman numerals

There is a further mystery: On the page proofs there are 3 Roman numerals, not in the handwritings of Hilbert nor of his wive Käthe. Furthermore, Hilbert practically never used Roman numerals. If he had to renumber, he used Arabic numerals in a different color, Roman numerals for numerating volumes, chapters or theorems only.

In the accompanying letter to Klein, Hilbert has written^[44]: "Hereby, I send you the first proofs * (3 sheets) of my first communication, (* Please, kindly to return them to me, because I have no other records.)"

Knowing this passage, the forger had the problem what Hilbert could have meant with "3 sheets". Even to him, it seemed incredible, Hilbert would have denoted page 1-2, the large sheet two (i.e. a double page with pages 6-3 and 4-5) and the mutilated page 7-8, though all three have a different size, collectively as "sheets".

Therefore, the forger added Roman numerals. The Roman number III is on page 7 directly beneath the cut-off. (The printed number 7 had disappeared because of the cut-off.) Thus, seemingly, he could prove the Roman numbers have been written after the cut-off, or in other words the cut-off was done before posting the letter (Mar 7, 1918).

The forger, though very cunning without any doubt, here again has betrayed himself: Even while assuming (already excluded above) Hilbert had sent Klein the above mentioned variously sized "3 sheets" (i.e. pages 1 to 8), it is still completely incomprehensible Hilbert would have added these Roman numerals. The consignment already had the original printed pagination, except as usual the title page 1 and the mutilated page 7-8. Hilbert simply had written a 7 and 8 beneath the

cut-off. The Roman numerals make sense only because the forger was uncomfortable selling such different formats as the "3 sheets" mentioned by Hilbert.

The Roman numerals are a fact which does not have a reasonable explanation when everything has gone the right way. However, they find a reasonable explanation by the motivation of a forger.

8.5 Crossing out pages 7 and 8 with a pencil

Now we arrive at a further mystery: The remaining 2/3 of the page 7-8 (i.e. later than the cut-off) have been crossed through from top left to bottom right with a pencil, again erased later on, but still slightly visible^[45].

These pencil vestiges are a fact, which do not have a reasonable explanation if everything has gone the right way. Plausible explanation for a forger: Soon after the cut-off (i.e. before the foldings) the forger had crossed out the remaining page 7-8, suggesting Hilbert is considering the page incorrect, making it plausible he had cut off something to save a bit of time. Later the forger had a better idea of the foldings (because they would date the cut-off before Mar 7, 1918). Now the cross out is obsolete, since Hilbert would not cross out what he sends Klein. Also the forger might have realized Hilbert would have crossed out the whole pages 7-8 to mark them as obsolete, and eventually later on he had cut-off something from these wrong pages in order to save time. After the cut-off the pages were sufficiently marked as irrelevant and there were no need of an additional crossing-out.

8.6 Wrong Archive-pagination

One more mystery: Old handwritings or valuable printed documents (such as Hilbert's page proofs) coming new into the Göttingen archive, are paginated by an archivist in the middle of the right edge using a pencil. In the following we will call it 'Archive-pagination' or 'A-pagination' for short. As a rule, this A-pagination is done with great care. Already originally (i.e. print) paginated documents (such as Hilbert's page proofs, pages 1-13), as a rule, are not paginated again.

It is a mystery Hilbert's page proofs have been paginated at all, and in a completely wrong order, especially astonishing since an original printed pagination was present. The A-pagination on the mutilated page was done at the right edge exactly in the middle of the remaining 2/3 of the page, making it clear - apparently - the A-pagination was done after the cut-off.

In 1997 Corry phones the archive, reporting the completely wrong A-pagination (wrong order). The archive rubs out the old (wrong) A-pagination and replaces it by a correct A-pagination. With a magnifying glass the old A-pagination can still be recognized.

The mystery of the wrong A-pagination finds a plausible explanation by supposing the page proofs have not been paginated while they arrived into the archive, since they had already the original printed pagination. After the forger had done the cut-off, had ripped-apart the second sheet, he added an A-pagination. This should prove the page proofs already had the cut-off while they arrived into the archive. The intentional wrong A-pagination makes sense, because a new A-pagination must be done, so the handwriting of the forger disappears.

8.7 Forgery of a date and a dog-ear in Hilbert's private notes

As we have seen in subsection (6.2) the sheet with archive page number 32 in Hilbert's folder "*Über Elektrodynamik*" are his notes while he has found the correct coefficients $\alpha = 1$, $\beta = -1/2$ and $\lambda = 0$ in the explicit field equations. As can be seen on a facsimile in Wuensch^[21] p. 60, this sheet 32 on top left bears the date 9/10.IX.15. It is certainly a good habit to add a date in all our own private notices. But had Hilbert had that habit? At no other sheet of his folder we can find such a date. Only with poor graphological capabilities one would suspect that the date has not the same handwriting as the remainder of the page, especially since both should be written in the same minute, with the same quill and with the same ink, and especially comparing it with the facsimile (Wuensch, p. 22) of a date on Hilbert's letter to Klein. A possible motivation for a forger would be to prove Hilbert has found or verified the explicit field equations only after he has seen them in Einstein's publications.

But the curiosities continue. I have ordered a copy of Hilbert's folder from the Göttingen archive^[12] and I was struck the date is now absent. From Göttingen I got the answer a turned-down-corner (dog ear) at the top left of Hilbert's page 32 sheet has prevented the date to become visible on the microfilm from which all requested copies are now made. Coincidence or an attempt by anyone who is interested to reduce the number of too obvious and thus counterproductive forgeries?

8.8 How is it possible to manipulate documents in the Göttingen archive?

For the question how it is possible to smuggle documents out of the Göttingen archive and to manipulate them alone, I refer to Wuensch's book^[21]. For instance it is possible to put them between one's own sheets, because while leaving the archive for a rest, personal working material is not checked.

8.9 What was on the cut-off in Hilbert's page proofs

There is agreement among historians the cut-off contained equation

$$H = K + L \tag{41}$$

where H is the total Lagrangian density, and K was defined on the cut-off e.g. as 'the invariant stemming from Riemann's tensor', i.e. $K \equiv R = \text{Ricci-scalar}$.

Some historians believe the cut-off also contained the explicit field equations (5c). Though we cannot exclude this possibility definitely, there are several arguments (see also T. Sauer^[47]) that this is improbable:

The cut-off is rather short, for all the definitions and explanations. Furthermore, the subject just before the cut-off is different, thus some transitional phrasing would be required. To give the explicit field equations does not fit to the position of the cut-off but rather after Eq. (26) of the page proofs, a position where they appeared in the printed version.

In the whole article (page proofs, published version and republished version) Hilbert was mainly interested in general theorems and in the energy concept. Hilbert did not include explicit formulae for the electromagnetic field equations either.

The most decisive argument, however, is the following: the editors of *Science* would not have passed the 1997-article without seeing a copy of the page proofs. They had observed the cut-off, since the main statement of the paper is, that nowhere in the whole page proofs the explicit field equations can be found. Thus, we can conclude that at the discovery of the page proofs in the Göttingen archive short after 1994, the cut-off was not yet there. Seeing the complete page-proofs, the editors had recognized the explicit field equations, even if formulated in a version like (5c), different from a modern one. Thus we can conclude the cut-off did not contain the explicit field equations.

8.10 What was the motivation of the forger?

Some historians believe the cut-off on Hilbert's page proofs contained the explicit field equations. Then the motivation would be to withdraw Hilbert's priority to have found the explicit field equations before Einstein, and to be able to publish this new historical perspective.

In the above subsection we have collected arguments the cut-off did not contain the explicit field equations. So what could then be the motivation for a forger to perpetrate the cut-off?

One strange uttered hypothesis^[46] is that self appointed advocates of Hilbert had done the cut-off in order at least to have a chance to speculate Hilbert has found the field equations before Einstein.

But there is a more serious and very plausible motivation: For a historian of science, it is a great triumph to be able to reverse the established opinion about the priority of a milestone in scientific progress, as was the final formulation of general relativity. After having detected the page proofs did not contain the explicit field equations, this success of that historian was jeopardized by the argument Hilbert was nevertheless the first to have given the Ricci-scalar as the correct Lagrangian density for the gravitational field, and the evaluation of the corresponding explicit field equations was merely a straightforward technical exercise.

The human psychology reacts more to a fast change of possession than to its actual level. Thus in a fit of ill-considered activity the historian, or someone else enjoyed by that initial success, made the cut-off by removing with a razor blade Hilbert's definition of K as the Ricci-scalar. Thus he hoped to argue that Hilbert has made his choice for K only while preparing for his printed version and he has added the definition of K as the Ricci-scalar only then, together with the corresponding explicit field equations. And that Hilbert had made the cut-off himself in order to paste some formulae somewhere else in order to save time.

At no other place on the remaining page proofs it can be decided if Hilbert had really taken the Ricci scalar, or if he had left his Lagrangian density unspecified. The only remaining hint is Hilbert's choice of the letter K, which is similar to k which was used by Gauss for curvature, see (11). Even that is now questioned by T. Sauer^[47], footnote 7, arguing Hilbert has taken K for alphabetical reasons. Except that hint, thus far, nobody doubts Hilbert had indeed defined on the cut-off his Lagrangian density K as 'the invariant stemming from Riemann's tensor', i.e the Ricci scalar, as can be seen by the printed version. However, we are not in need to argue the forger was very anticipating.

8.11 Summary: a chronology of the forgery

After discovery of Hilberts intact page proofs in a Göttingen archive after 1994, it was observed they did not contain the correct explicit field equations of General Relativity. A publication of this spectacular historical fact, reversing the priority dispute in favor of Einstein, was begun.

Shortly, by internal discussions, it was argued Hilbert still deserves the priority of having found the correct Lagrangian density, and the evaluation of the explicit field equations could be considered a technical computational exercise only.

Therefore, a forger, who must have detailed insight in the relevant physics and mathematics, made the shortest possible cut-off, so that to day it is no longer possible to prove with the remaining part of the page proofs Hilbert had taken the Ricci scalar as the Lagrangian density. One could now argue Hilbert had inserted not only the correct field equations but also the correct Lagrangian density only after Dec 6, 1915. intention to prove the cut-off was done in Hilbert's time. First with a pencil, he crossed out the remaining two-thirds of pages 7 and 8, what he rubbed out later. These traces are a fact on the page proofs, which do not have a reasonable explanation if everything had gone the right way. However, they find a plausible explanation by the motivation of a forger. One should assume Hilbert had made the crossings indicating the contents of the pages are obsolete. So it becomes more probable Hilbert has made the cut-off himself to save time. Later, the forger recognized Hilbert would have done the crossing before the cut-off. After the cut-off the sheet was already sufficiently marked as unimportant. Therefore, the forger erased them again, removing this nonsensical action.

Only now the forger was thinking more seriously about decisive manipulations in order to prove Hilbert has done the cut-off himself. The forger must be in possession of relevant historical data, and in particular about the format and contents of Hilbert's letter to Klein^[44]. He asked himself: "How could I prove Hilbert has made the cut-off before sending the page proofs to Klein?". His answer: "I could fold the page proofs to the format of the accompanying letter to Klein, and if I fold the mutilated page in the remaining middle, I can prove the cut-off was done before sending the letter, i.e. before 1918." But the forger had a problem, because pages 1-2 (frontside-backside) are still connected with the mutilated pages 7-8 on the same large sheet. Therefore he separated page 1-2 from pages 7-8. Only then he was able to make the differing hand-foldings in their respective middle. In the vertical direction he made foldings to the format of the accompanying letter to Klein. But now he had a further problem, because in the accompanying letter Hilbert writes that he will send 3 sheets. The forger had three very different sheets: half a sheet (pages 1-2), an intact sheet (pages 3-4 and 5-6) and a mutilated sheet (pages 7-8). It seemed strange Hilbert had called them collectively as '3 sheets'. Therefore the forger added three Roman numerals to make that more plausible.

The forger observed Hilbert's page proofs had no archive pagination. This gave him the opportunity to add one with the pagination in the middle of the mutilated page. Thus he could prove the cutoff was already there when the page proofs came into the archive, which is the time the archive pagination is usually done.

9 Acknowledgements

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- [10] CRS's revolutionary new historical discovery was also published in several daily newspapers, including THE WASHINGTON POST by giving interviews. See: Curt Suplee, Researchers Definitely Rule Einstein Did Not Plagiarize Relativity Theory, THE WASHINGTON POST, Nov 14, 1997, page A24.
- [11] Felix Klein, Hilbert's 13 years older colleague in Göttingen, had asked Vermeil in 1918, to explain him in detail Hilbert's method. Vermeil proposed a tricky procedure. It is described in detail by Wuensch^[21] p. 65. Vermeil expressively calls Hilbert's [our box Eq. (1)] an assumption. Thus Vermeil was aware, that it was not a proved fact, as Hilbert had insinuated.
- [12] Copies can be ordered from the Göttingen Archive by e-mails to: hsd@mail.sub.unigoettingen.de. Hilbert's 'special case' is mentioned in his private folder 622 entitled Zur Elektrodynamik.
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