Numerical methods MATHMODES course:
Exercise sheet 1

1. Solve the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $(W_t)_{t\geq 0}$ is a real-valued Brownian motion, and $\mu, \sigma \in \mathbb{R}$ are fixed constants (the *drift* and *volatility* respectively). This is the Black-Scholes model of the price of a stock at time t, or equivalently a geometric Brownian motion. Show that the mean and variance of S_t are given by

$$\mathbb{E}(S_t|S_0) = S_0 e^{\mu t}, \quad \text{and} \quad \mathrm{Var}(S_t|S_0) = S_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1),$$

respectively.

2. Suppose that $(S_t)_{t\geq 0}$ is defined as in Exercise 1. Find the stochastic differential equations satisfied by

(a) $Y_t = \cos(S_t)$;

(b) $Y_t = t^2\sqrt{S_t}$;

(c) $Y_t = S_t^n$ for some constant n, and hence deduce that S_t^n is also a geometric Brownian motion.

3. Suppose that $(S_t)_{t\geq 0}$ is defined as in Exercise 1, where it represents the USD/YEN exchange rate at time t, and that today's exchange rate is S_0. You are told that both the expected USD/YEN exchange rate (given by S_t) and the YEN/USD exchange rate (given by $1/S_t$) a year ahead are $2S_0$. Is this possible, and if so how? (Hint: write down the SDEs satisfied for both S_t and $1/S_t$ using part (c) above, and then use Exercise 1.)

4. Prove Gronwall's Lemma: Suppose $g : [0,T] \to \mathbb{R}$ and that there exist positive integrable functions $\alpha, \beta : [0,T] \to \mathbb{R}$ such that

$$g(t) \leq \alpha(t) + \int_0^t \beta(s) g(s) ds, \qquad \forall t \in [0,T].$$

Then

$$g(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s)e^{\int_s^t \beta(r)dr}ds, \qquad \forall t \in [0,T].$$

In particular, if α is non-decreasing show that

$$g(t) \leq \alpha(t)e^{\int_0^t \beta(s)ds}, \qquad \forall t \in [0,T].$$

Hint: Define

$$f(s) := \exp\left(-\int_0^s \beta(r)dr\right)\int_0^s g(r)\beta(r)dr, \qquad s \in [0,T],$$

and look at $f'(s)$ to get a bound for $\int_0^t g(s)\beta(s)ds$.

5. Simulate paths of $(S_t)_{t\geq 0}$ given in Exercise 1 with $S_0 = 1$ using the Euler method in the following different cases (and display all three results on the same figure):

(a) $\mu = 1$, $\sigma = 0.2$;

(b) $\mu = 1$, $\sigma = 0.5$;

(c) $\mu = 2$, $\sigma = 0.2$.

Hence observe the effect of changing the volatility σ and the drift μ on the path $(S_t)_{t\geq 0}$. Change your code to repeat the exercise with the Milstein scheme.

6*. Consider the Cox-Ingersoll-Ross (CIR) process $(R_t)_{t\geq 0}$ given by

$$dR_t = \theta(\mu - R_t)dt + \sigma\sqrt{R_t}dW_t, \qquad R_0 = r_0 > 0,$$

where $(W_t)_{t\geq 0}$ is a real-valued Brownian motion and θ, μ and σ are fixed positive constants. Deduce that $R_t \geq 0$ for all $t \geq 0$.

ASIDE: This fact makes the CIR process useful in modeling interest rates (which are typically non-negative). In fact the model can also handle the situation where we want to model the interest rate by a *strictly* positive process. Indeed, it can be shown that if $2\theta\mu \geq \sigma^2$ then $R_t > 0$ for all $t \geq 0$ almost surely.

The SDE satisfied by the CIR process is an example of an SDE that cannot be integrated i.e. there is no explicit solution for $(R_t)_{t\geq 0}$ as a function of $(W_t)_{t\geq 0}$. We therefore would like to simulate paths of this process. What is the problem of using an Euler scheme to simulate R_t? Can you suggest a way to improve this scheme?

Reference:
`http://cermics.enpc.fr/reports/CERMICS-2005/CERMICS-2005-279.pdf.`

Homework

Exercise 1

$$dS_t = \mu S_t dt + \sigma S_t dW_t \qquad \text{GBM}$$

$$df = f_1 dt + f_2 dx + \frac{1}{2}\left[f_{11} dt^2 + f_{22} dx^2 + f_{12} dt\, dx \right]$$

$$f = \ln x \qquad f_1 = 0 \qquad f_2 = \frac{1}{x} \qquad f_{22} = -\frac{1}{x^2}$$

$$df(t, S_t) = 0 + \frac{1}{S_t} dS_t + \frac{1}{2}\left[-\frac{1}{S_t^2}(dS_t)^2 \right] \qquad dS_t^2 = \mu^2 S_t^2 dt^2 + \sigma^2 S_t^2 dt + \mu\sigma S_t^2 dt\, d\ldots$$

$$= \frac{1}{S_t}\left[\mu S_t dt + \sigma S_t dB_t \right] - \frac{1}{2S_t^2}\left[\sigma^2 S_t^2 dt \right]$$

$$= \mu dt + \sigma dB_t - \frac{1}{2}\sigma^2 dt.$$

$$d\ln(S_t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t.$$

$$\ln(S_t) = \ln(0, S_0) + \ldots$$

$$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t \right]$$

Expectation

$$\mathbb{E}[S_t] = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t\right] \underbrace{\mathbb{E}\left[e^{\sigma B_t}\right]}_{\text{Momentum Generating function of } N(0,t)}$$

$$e^{\sigma 0 + \frac{1}{2}\sigma^2 t^2}$$

$$\mathbb{E}[S_t] = S_0 e^{\mu t}$$

$$e^{\mu\lambda + \frac{1}{2}\lambda^2 \underbrace{\sigma^2}_{\text{variance}}}$$

variance

$$\mathbb{E} S_t^2 - (\mathbb{E} S_t)^2$$

$$\mathbb{E}\, S_t^2 = \mathbb{E}\left[S_0^2\, e^{2(\mu - \frac{1}{2}\sigma^2)t + 2\sigma B_t} \right]$$

$$= S_0^2\, e^{2(\mu - \frac{1}{2}\sigma^2)t}\, \underbrace{\mathbb{E}\left[e^{2\sigma B_t} \right]}$$

momentum generator function $N(0,t)$

$$e^{2\sigma 0 + \frac{1}{2} t (2\sigma)^2}$$

$$= S_0^2\, e^{2\mu t - \sigma^2 t + 2\sigma^2 t} = S_0^2\, e^{2\mu t + \sigma^2 t}.$$

variance

$$\operatorname{var} S_t = S_0^2\, e^{2\mu t + \sigma^2 t} - \left(S_0\, e^{\mu t} \right)^2 = S_0^2\, e^{2\mu t}\left[e^{\sigma^2 t} - 1 \right]$$

Exercise 2

$Y_t = \cos S_t$. $\quad Y_1 = 0 \quad Y_2 = -\sin S_t \quad Y_{22} = -\cos S_t.$

$$dY = \left(Y_1 + \frac{1}{2} Y_{22} \right) dt + Y_2\, dS_t.$$

$$= -\frac{\cos S_t}{2}\, dt + \sin S_t \left[\mu S_t\, dt + \sigma S_t\, dB_t \right]$$

$$= \left(-\mu S_t^{\sin S_t} - \sigma^2 S_t^2 \frac{\cos S_t}{2} \right) dt + \left[\sigma S_t \sin S_t\, dB_t \right]$$

$f = S_t^n \qquad f(t,x) = x^n$

$f_1 = 0 \quad f_2 = n x^{n-1} \quad f_{22} = n(n-1) x^{n-2}$

$$df = f_1 dt + f_2 dx + \frac{1}{2}\left[f_{11} dt^2 + f_{22} dx^2 + f_{12} dt\, dx \right]$$

$$= n x^{n-1} dx + \frac{1}{2} n(n-1) x^{n-2} dx^2$$

$$dS_t^n = n S_t^{n-1} \left[\mu S_t dt + \sigma S_t dB_t \right] + \frac{1}{2}\left[n(n-1) S_t^{n-2} \right]\left[\sigma^2 S_t^2 dt \right]$$

$$= \left(n S_t^{n} \mu + \frac{1}{2} n(n-1) S_t^n \sigma^2 \right) dt + n S_t^{n} \sigma\, dB_t .$$

$$= S_t^n \left[\mu n + \frac{\sigma^2}{2} n(n-1) \right] dt + S_t^n\, n\sigma\, dB_t .$$

GBM with $\mu' = n\mu + \frac{1}{2} n(n-1)\sigma^2$

$\sigma' = n\sigma .$

powers of S_t^n is also a GBM.

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

$$dS_t^{-1} = S_t^{-1}\left[-\mu + \frac{\sigma^2}{\cancel{2}} \cancel{2} \right] dt + S_t^{-1}(-\sigma\, dB_t)$$

$\mathbb{E} S_t = S_0 e^{\mu t} \qquad \mathbb{E} S_t^{-1} = S_0^{-1} e^{(-\mu + \sigma^2)t}$

$t = 1 \quad = 2S_0 \qquad t = 1 \quad = 2S_0$

$e^{\mu} = 2 \qquad e^{-\mu + \sigma^2} = 2 .$

$\Rightarrow e^{\sigma^2} = 4 , \quad \sigma^2 = \ln 4 \; ; \quad \mu = \ln 2 .$

IS POSSIBLE that A RATE and ITS INVERSE ARE SAME!! BAD MODEL!!!

Exercise 3 Gronwall Lemma

if $g(t)$ satisfy an differential inequality
or integral

then $g(t)$ is bounded by the solution of the inequality.
(doesnt have $g(t)$ in both sides)

Inequality $\quad g(t) \leq \alpha(t) + \int_0^t \beta(s)\, g(s)\, ds \qquad \exists\ \alpha, \beta$ positive
(s.t if $g(t)$ moves up all RHS moves up.)

The solution $\quad f(t) = \int_0^t \beta(s)\, g(s)\, ds$

$$\frac{df}{dt} = \beta(t)\, g(t)$$

Inequality is $\quad \frac{f'}{\beta} \leq \alpha + f \quad ; \quad f' \leq \alpha\beta + f\beta$

$f' - f\beta(t) \leq \alpha\beta \qquad$ integrating factor $\quad I = e^{-\int_0^t \beta(r)\, dr}$

Fund Thm

$$\frac{dIf}{dt} \leq \alpha\beta I \qquad \frac{d}{dt}\left[\int_0^t \beta(r)\, dr\right] = \frac{d}{dt}\left[F(t) - F(0)\right] = \beta(t)$$

$$If - I_0 f(0) \leq \int_0^t \alpha\beta I\, ds$$

$$f \leq I^{-1} \int_0^t \alpha(s)\, \beta(s)\, e^{-\int_0^s \beta(r)\, dr}\, ds$$

$$f(t) \leq e^{+\int_0^t \beta(r)\, dr} \int_0^t \alpha(s)\, \beta(s)\, e^{-\int_0^s \beta(r)\, dr}\, ds$$

$$f(t) \leq \int_0^t \alpha(s)\, \beta(s)\, e^{\int_s^t \beta(r)\, dr}\, ds$$

Now add smt, positive [illegible] [illegible] because $f(t) = \int_0^t \beta(s)\, g(s)\, ds$

$$g(t) \leq \alpha(t) + f(t)$$

$$g(t) - \alpha(t) \leq f(t)$$

$$g(t) - \alpha(t) \leq \int_0^t \alpha(s)\beta(s)\, e^{\int_s^t \beta(r)\,dr}\, ds$$

$$g(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s)\, e^{\int_s^t \beta(r)\,dr}\, ds.$$