Aylin

Université de Nice - M2 Mathmods — Stochastic Calculus - 2014/2015

WORKSHEET 1

2, 4, 11, 12, 14

In all the exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes the current probability space.

1. Random Variables

Exercise 1. Compute $\mathbb{V}(X)$ (if exists) in the following cases

(1) X is a r.v. of uniform law on $(0,1)$.
(2) X is a r.v. of Bernoulli law of parameter $p \in (0,1)$.
(3) X is a r.v. of Gaussian law $\mathcal{N}(m, \sigma^2)$ (i.e. of parameters $m \in \mathbb{R}$ and $\sigma > 0$).

Exercise 2. Let X be a r.v. of Gaussian law $\mathcal{N}(0,1)$ (i.e. of parameters 0 and 1).

(1) For $m \in \mathbb{R}$ and $\sigma > 0$, give the law of $m + \sigma X$.
(2) Give the law of X^2.

$\mathcal{N}(m, \sigma^2)$

Exercise 3. Let U be a r.v. of uniform law on $(0,1)$. Give the law of $1 - U$.

Exercise 4. Let X be a r.v. and $t \in \mathbb{R}$. Compute $\mathbb{E}[\exp(tX)]$ in the following cases:

(1) X is a Bernoulli r.v. of parameter $0 < p < 1$.
(2) X is a Gaussian r.v. $\mathcal{N}(m, \sigma^2)$, $m \in \mathbb{R}$ and $\sigma > 0$.

Exercise 5. Show that the moments of a r.v. X of Gaussian law $\mathcal{N}(0,1)$ are given by

$$\forall n \geq 0, \ \mathbb{E}(X^{2n}) = \frac{(2n)!}{2^n n!}, \ \mathbb{E}(X^{2n+1}) = 0.$$

Hint: Use the previous exercise.

2. Independent Random Variables

Exercise 6. Let X, Y be two independent and identically distributed r.v. of law $\mathcal{N}(0,1)$. Prove that $X - Y$ and $X + Y$ are independent. split f_{X+Y}

Exercise 7. Let U and V be two independent and identically distributed r.v. of uniform law on $(0,1)$. What is the law of $\max(U,V)$? What is the law of the pair $(\min(U,V), \max(U,V))$?

Exercise 8. Let X and Y be two independent and identically distributed r.v. of Gaussian law $\mathcal{N}(0,1)$. What is the law of X/Y? Is it possible to define $\mathbb{E}[X/Y]$? X/Y

Exercise 9. Let X and Y be two independent variables such that $\mathbb{E}[X^2 + Y^2] < +\infty$.

(1) Show that $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ exist.
(2) Show that $\mathbb{V}(X+Y) = \mathbb{V}(X) + \mathbb{V}(Y)$.
(3) Find a counterexample to the above equality when X and Y are not independent.

Exercise 10. Let U and V be two independent exponential r.v. of parameter $\lambda > 0$. What is the law of $\min(U,V)$?

Exercise 11. Let X and Y be two independent Gaussian random variables of law $\mathcal{N}(m_1, \sigma_1^2)$ and $\mathcal{N}(m_2, \sigma_2^2)$, $m_1, m_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$. Using characteristic functions, give the law of $X_1 + X_2$.

X + Y

3. Gaussian vectors

Exercise 12. Let $m = (m_i)_{1 \le i \le n} \in \mathbb{R}^n$ and $K = (K_{i,j})_{1 \le i,j \le n}$ be a non-negative symmetric matrix. What is the law of $m + K^{1/2}(X_1, \dots, X_n)^t$, where $X_1, \dots, X_n$ are n I.I.D. random variables of $\mathcal{N}(0, 1)$ law?

Exercise 13. Let $(X_1, \dots, X_n)$ be a Gaussian vector and $(i_1, \dots, i_m) \in \{1, \dots, n\}^m$. What can be said about the law of $(X_{i_1}, \dots, X_{i_m})$?

Exercise 14. Let X be an $\mathcal{N}(0, 1)$ r.v. and Z be a uniformly distributed r.v. on $\{-1, 1\}$, independent of X.

(1) Show that ZX is Gaussian.
(2) Considering $X + ZX$, show that the pair (X, ZX) isn't Gaussian.
(3) Prove that X and ZX are not independent, but that their covariance is zero.

Exercise 15. Let $X_1, \dots, X_n$ be n Gaussian independent r.v. Check that the sum $\sum_{i=1}^n X_i$ is a Gaussian r.v. whose mean and variance are respectively given by the sum of the means and the sum of the variances of the $(X_i)_{1 \le i \le n}$.

Exercise 16. Let $(X_1, \dots, X_n)$ be a Gaussian random vector with mean $m = (m_j)_{1 \le j \le n}$ and covariance matrix $K = (K_{j,k})_{1 \le j,k \le n}$.

(1) For some $(t_j)_{1 \le j \le n} \in \mathbb{R}^n$, what is the law of $\sum_{j=1}^n t_j X_j$?
(2) Deduce that

$$\mathbb{E}\big[\exp\big(i \sum_{j=1}^n t_j X_j\big)\big] = \exp\big(i \sum_{j=1}^n t_j m_j - \frac{1}{2} \sum_{j,k=1}^n t_j K_{j,k} t_k\big)\big].$$

(3) What can be said about two Gaussian vectors with the same mean and the same covariance?

Exercise 17. Let $(X_1, \dots, X_m)$ and $(Y_1, \dots, Y_n)$ be two Gaussian vectors such that **the vector** $(X_1, \dots, X_m, Y_1, \dots, Y_n)$ **is Gaussian**. Show that $(X_1, \dots, X_m)$ and $(Y_1, \dots, Y_n)$ are independent if and only if the covariance matrix of $(X_1, \dots, X_m, Y_1, \dots, Y_n)$ is block diagonal, i.e. has the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ where A is a $m \times m$ square matrix and B is a $n \times n$ square matrix.

Exercise 18. Let $(X_i)_{1 \le i \le n}$, $n \ge 2$, be n independent and identically distributed r.v. of Gaussian law $\mathcal{N}(0, 1)$. Prove that the r.v. $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ and $\max_{1 \le i \le n} X_i - \min_{1 \le i \le n} X_i$ are independent.
Hint: Consider the vector $(\bar{X}_n, X_1 - \bar{X}_n, \dots, X_n - \bar{X}_n)^t$.

Exercise 19. Let $(X_n)_{n \ge 1}$ be a sequence of I.I.D. r.v. of Gaussian law $\mathcal{N}(0, 1)$. We set $B_0 = 0$ and for $n \ge 1$, $B_n = \sum_{k=1}^n X_k$.

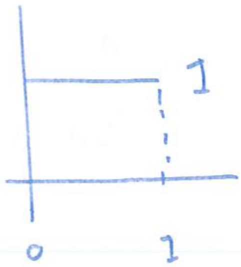
(1) Give the covariance matrix of $(B_1, \dots, B_n)$ as well as its probability density (if exists).
(2) For $1 \le m \le n$, set $Z_m = B_m - (m/n)B_n$. Prove that Z_m and B_n are independent.

(Above, the first diagonal block is of size $m \times m$ and the second one of size $n \times n$.

Worksheet 1

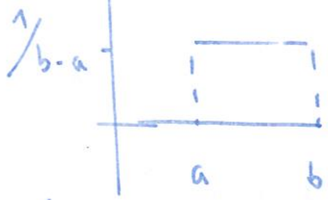
Exercise 1.

a) $X \sim U[0,1]$



$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 \end{cases}$

$\mu = E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$



$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 \end{cases}$

$= \int_0^1 x\, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

$Var(X) = E[(X-\mu)^2] = E[X^2] - \mu^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$

$= \int_0^1 x^2\, dx - \mu^2 = \frac{x^3}{3}\Big|_0^1 - \mu^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$

b) $Ber(p)$ $\quad p \in (0,1)$ $\quad X = \begin{cases} 1 & \text{sucess} \\ 0 & \text{failure} \end{cases}$

1 Trial $\quad f(x) = p^x (1-p)^{1-x}$

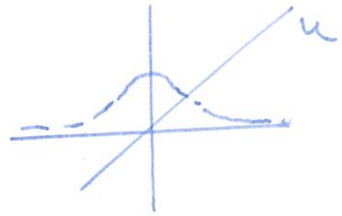
n Trial $\quad f(x) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$

Binomial Expansion $\quad (q+p)^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$

$\mu = E[X] = \sum_x x\, f(x) = 1 f(1) + 0 f(0) = p \quad$ (1 trial)

$Var(x) = E[X^2] - \mu^2 = \sum_x x^2 f(x) - p^2 = p(1-p) \quad$ (1 trial

c) $X \sim N(\mu, \sigma^2)$



$$\mu = E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx =$$

$$u = \frac{x-\mu}{\sigma} \qquad du = \frac{dx}{\sigma}$$

$$= \int_{-\infty}^{\infty} (\mu + \sigma u) \frac{1}{\sigma\sqrt{2\pi}} e^{-u^2/2} \sigma \, du$$

$$= \int_{-\infty}^{\infty} \mu \frac{1}{\sqrt{2\pi}\,\sigma} e^{-u^2/2} \sigma \, du + \underbrace{\int_{-\infty}^{\infty} \sigma u \frac{1}{\sigma\sqrt{2\pi}} e^{-u^2/2} \sigma \, du}_{=0}$$

$$= \mu + 0 = \mu \;//$$

$$Var(X) = E[X^2] - \mu^2 = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx - \mu^2$$

$$= \int_{-\infty}^{\infty} (\mu + \sigma u)^2 \frac{1}{\sqrt{2\pi}\,\sigma} e^{-u^2} \sigma \, du - \mu^2$$

$$= \int_{-\infty}^{\infty} (\mu^2 + \sigma^2 u^2 + 2\mu\sigma u) \frac{1}{\sqrt{2\pi}\,\sigma} e^{-u^2/2} \sigma \, du - \mu^2$$

$$= \frac{1}{\sqrt{2\pi}} \sigma^2 \int_{-\infty}^{\infty} u^2 \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du = \sigma^2 \left[\frac{\cancel{x}^2}{-\cancel{x}} e^{-u^2/2} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{e^{u^2/2}}{x} \} x \, d \right.$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \left[0 + \sqrt{2\pi} \right] = \sigma^2$$

Exercise 2.

$X \sim N(0, I)$

1) Law of $m + \sigma X = u$

$u = \phi(x)$

$x = \phi^{-1}(u)$ $\qquad dx = [\phi^{-1}(u)]' du$

$$P(u_1 \leq U \leq u_2) = P(\phi(x_1) \leq \phi(X) \leq \phi(x_2))$$

$$= P(x_1 \leq X \leq x_2)$$

$$\int_{u_1}^{u_2} g(u)\, du = \int_{x_1}^{x_2} f(x)\, dx$$

$$= \int_{u_1}^{u_2} f(\phi^{-1}(u)) \, [\phi^{-1}(u)]' \, du$$

as $x = \dfrac{u-m}{\sigma} = \phi^{-1}(u)$

$$= \int_{u_1}^{u_2} f\left(\frac{u-m}{\sigma}\right) \frac{1}{\sigma}\, du$$

$$\boxed{g(u) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(u-m)^2}{2\sigma^2}}}$$

Exercise 3

$X \sim U[0,1]$ $\quad u = \phi(x)$

$u \sim 1 - \cancel{X}$ $\quad x = \phi^{-1}(u)$

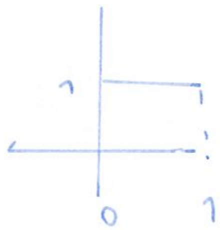
$X = 1 - u = \phi^{-1}(u)$

$$\int_{u_1}^{u_2} g(u)\,du = \int_{x_1}^{x_2} f(x)\,dx$$

$$= \int_{u_1}^{u_2} f(\phi^{-1}(u)) \frac{d}{du}\phi^{-1}(u)\,du.$$

$$= \int_{u_1}^{u_2} f(1-u)(-1)\,du = \int_{u_2}^{u_1} f(1-u)\,du \quad *$$

$f(x) =$



$$\Rightarrow f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \end{cases}$$

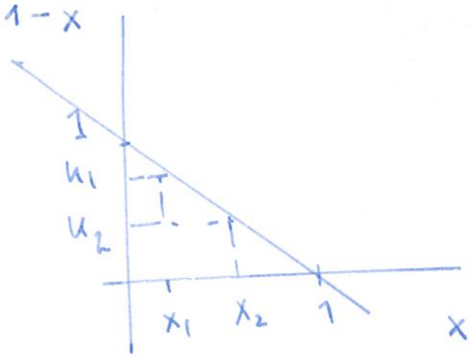
$$f(1-u) = \begin{cases} 1 & 0 \le 1-u \le 1 \\ 0 & \end{cases}$$

$0 \le 1 - u \le 1$

$-1 \le -u \le 0$

$1 \ge u \ge 0$

$$f(1-u) = \begin{cases} 1 & 0 \le u \le 1 \\ 0 & \end{cases}$$



$$g(u) = \begin{cases} 1 & 0 \le u \le 1 \\ 0 & \end{cases}$$

$x_2 \ge x_1$

$u_2 \le u_1$ $\quad *$

$$\boxed{g(u) = f(x)}$$

Exercise 5

$$N(m, \sigma^2) \qquad E[e^{rX}] = e^{\mu r + \frac{\sigma^2 r^2}{2}}$$

$N(0,1)$

$$E[e^{\tau X}] = e^{\tau^2/2} = \sum_{n=0}^{\infty} \frac{\tau^{2n}}{2^n n!}$$

$\cancel{E[X^{2n}]} =$

$$E[e^{\tau X}] = E\left[\sum_{n=0}^{\infty} \frac{(\tau X)^n}{n!}\right] = E\left[\sum_{n=0}^{\infty} \frac{(\tau X)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(\tau X)^{2n+1}}{(2n+1)!}\right]$$

$$= E\left[\sum_{n=0}^{\infty} \frac{\tau^{2n} X^{2n}}{(2n)!}\right] + \underbrace{E\left[\sum_{n=0}^{\infty} \frac{(X\tau)^{2n+1}}{(2n+1)!}\right]}_{=0}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{\tau^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{\tau^{2n}}{(2n)!} E[X^{2n}]$$

$$E[X^{2n}] = \frac{(2n)!}{2^n n!}$$

Exercise 6

$X, Y \sim N(0,1)$

Gaussian ind $\iff$ uncorrelated

$$\operatorname{cov}(X+Y, X-Y) = E[(X+Y)(X-Y)]$$
$$= E[X^2] - [Y^2] = 1 - 1 = 0.$$

Exercise 8

X, Y iid $N(0,1)$ Law $\frac{X}{Y}$

$U = \frac{X}{Y}$ arbitrarily $V = Y$ $\Rightarrow$ $\begin{matrix} x = uv \\ y = v \end{matrix}$ $\quad |J| = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$

$$g(u) = \int_{-\infty}^{\infty} |v| \, f(uv, v) \, dv = \frac{1}{2\pi} \int_{-\infty}^{\infty} |v| \, e^{-(uv)^2/2} e^{-v^2/2} \, dv$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |v| \, e^{-v^2 \underbrace{\left(\frac{1}{2} + \frac{u^2}{2}\right)}_{a}} dv = \frac{2}{2\pi} \int_{0}^{\infty} v \, e^{-v^2 a} \, dv$$

$j = av^2$
$dj = 2av\,dv$
$$= \frac{2}{2\pi} \int_{0}^{\infty} \frac{1}{2a} e^{-j} \, dj = \frac{1}{2a} \left. \frac{e^{-j}}{-1} \right|_{0}^{\infty} = \frac{1}{2a}$$

$$\Rightarrow \quad g\left(\frac{X}{Y}\right) = \frac{1}{2\left(\frac{1}{2} + \frac{u^2}{2}\right)} = \frac{1}{\left(1 + \frac{X}{Y}\right)^2}$$

Cauchy distribution or Lorentzian

Exercise 9 $\quad E[X^2+Y^2] < \infty$

Cauchy Schwarz $\quad |\langle u,v\rangle| \leq \|u\|_{L^2} \, \|v\|_{L^2}$

$$E[|XY|] \leq \left(E[X^2]\right)^{1/2}\left(E[Y^2]\right)^{1/2}$$

$$\|XY\|_{L^1} \leq \|X\|_{L^2} \, \|Y\|_{L^2}$$

1) if $Y=1 \Rightarrow \quad E[X] \overset{CS}{\leq} \left(E[X^2]\right)^{1/2} \leq \left(E[X^2+Y^2]\right)^{1/2} < \infty$

same for $E[Y]$

2) $\operatorname{var}(X+Y) = \cancel{E[(X-\mu_x)(Y-\mu_y)]} \cancel{=} \cancel{E[XY] - \mu_x\mu_y}$

$$= E\left[\left((X+Y)-(\mu_x+\mu_y)\right)^2\right] = E\left[\left((X-\mu_x)+(Y-\mu_y)\right)^2\right]$$

$$= E\left[(X-\mu_y)^2\right] + E\left[(Y-\mu_y)^2\right] + E\left[2(X-\mu_x)(Y-\mu_y)\right]$$

$$= \operatorname{var} X + \operatorname{var} Y + 2\operatorname{cov}(X,Y)$$

[Exercise 1]

X, Y iid $N(m_i, \sigma_i^2)$

$$\Phi_X = E[e^{itX}] = e^{itm_1 - \frac{t^2\sigma_1^2}{2}}$$

$$\Phi_{X+Y} = \Phi_X \Phi_Y \quad (\text{iid})$$

$$= e^{itm_1 - \frac{t^2\sigma_1^2}{2}} \cdot e^{itm_2 - \frac{t^2\sigma_2^2}{2}}$$

$$= e^{it(m_1+m_2) - \frac{t^2}{2}(\sigma_1^2+\sigma_2^2)}$$

$$= N(m_1+m_2, \sigma_1^2+\sigma_2^2)$$

Exercise 12

$m = (m_i) \in \mathbb{R}^n$ K non negative symmetric $n \times n$ matrix

Law of $m + K^{1/2} X$ $X \in \mathbb{R}^n$ X_i are iid $N(0,1)$

$$y = m + K^{1/2} X \quad \text{should be } N(m, K)$$

Mean

$$E[y_i] = E[m_i + (K^{1/2} X)_i]$$
$$= E\left[m_i + \sum_j K^{1/2}_{ij} X_j\right] = m_i + \sum_j^n K^{1/2}_{ij} E[X_j]$$
$$= m_i$$
$$\Rightarrow E[y] = m$$

Covariance

$$E[(y_i - m_i)(y_j - m_j)] = E\left[\sum_k K^{1/2}_{ik} X_k \sum_l K^{1/2}_{jl} X_l\right]$$
$$= \sum_{k,l} K^{1/2}_{ik} K^{1/2}_{jl} \underbrace{E[X_k X_l]}_{\delta_{kl}}$$
$$= \sum_k K^{1/2}_{ik} K^{1/2}_{jk} = K^{1/2} K^{1/2\,T}$$
$$= K \quad \text{covariance matrix}$$

$$Y = m + K^{1/2} X \sim N(m, K)$$

Exercise 16

$T \in \mathbb{R}^n \quad X \in \mathbb{R}^n$

$$TX = \sum t_j X_j$$

$$E[TX] = E\left[\sum t_j X_j\right] = \sum_j t_j \, E[X_j] = \sum_j t_j m_j$$

$$= Tm$$

$$Var(TX) = \cancel{E\left[\left\|\sum_j t_j x_j\right\|^2\right]} = E\left[(TX - Tm)^2\right]$$

$$= E[TX]^2 - T^2 m^2$$

$$= E\left[\sum_j t_j X_j \sum_i t_i X_i\right] - T^2 m^2$$

$$\sum_{j,i} t_j \bar{t}_i \, E[X_i X_j] - T^2 m^2$$

$$= T \, cov(X_i, X_j) \, T^T$$

or

$$E\left[\left(\sum_i t_i X_i - \sum_i t_i m_i\right)\left(\sum_j t_j X_j - \sum_j t_j m_j\right)\right]$$

$$E\left[\sum_i t_i (X_i - m_i) \sum_j t_j (X_j - m_j)\right]$$

$$\sum_{ij} t_i t_j E\left[(X_i - m_i)(X_j - m_j)\right]$$

$$= T \, cov(X_i, X_j) \, T^T$$

Aylin

WORKSHEET 2

In all the exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes the current probability space.

1. Law of a Process

Exercise 1. Let $(X_t)_{0\leq t\leq 1}$ be a real-valued continuous process.

(1) Show that the following mapping is a random variable:

$$\omega \in \Omega \mapsto \int_0^1 X_s(\omega)ds.$$

Hint: think of Riemann sums.

(2) Let $(Y_t)_{0\leq t\leq 1}$ be another real-valued continuous process.

(a) Assume that X and Y have the same law, prove that $\int_0^1 X_s ds$ and $\int_0^1 Y_s ds$ have the same law.

(b) Assume that X and Y are independent, prove that $\int_0^1 X_s ds$ and $\int_0^1 Y_s ds$ are independent.

2. Gaussian Processes

Exercise 2. Let $(X_t)_{t\geq 0}$ be a Gaussian process. For a function ψ from $\mathbb{R}_+$ into itself, show that $(X_{\psi(t)})_{t\geq 0}$ is also Gaussian.

Exercise 3. Let $(X_t)_{0\leq t\leq 1}$ be a real-valued continuous Gaussian process. We suppose that the functions $t \mapsto \mathbb{E}(X_t)$ and $(t,s) \mapsto \mathbb{E}(X_s X_t)$ are continuous. Show that $\int_0^1 X_s ds$ has a Gaussian law. Compute its mean and its covariance.

3. Brownian Motion

Exercise 4. Let $(B_t)_{t\geq 0}$ be a (real) Brownian motion. Show that $(-B_t)_{t\geq 0}$ is a Brownian motion.

Exercise 5. Let $(B_t)_{t\geq 0}$ be a (real) Brownian motion. For a real $a > 0$, show that $(B_{a+t} - B_a)_{t\geq 0}$ is a Brownian motion and is independent of $(B_t)_{0\leq t\leq a}$.

Exercise 6. Let $(B_t)_{t\geq 0}$ be a (real) Brownian motion et and $(\widetilde{B}_t)_{t\geq 0}$ be the family of random variables given by:

$$\widetilde{B}_0 = 0, \ \forall t > 0, \ \widetilde{B}_t = tB_{t^{-1}}.$$

(1) Show that $(\widetilde{B}_t)_{t\geq 0}$ is a centered Gaussian process with $(s,t) \in \mathbb{R}_+^2 \mapsto s \wedge t$ as covariance function.

(2) Deduce that $(\widetilde{B}_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ have the same law.

Exercise 7. Let $(B_t)_{t\geq 0}$ be a (real) Brownian motion and $(Z_t)_{0\leq t\leq 1}$ be the process:

$$\forall t \in [0,1], \ Z_t = B_t - tB_1.$$

(1) Show that $(Z_t)_{0\leq t\leq 1}$ is a Gaussian process and is independent of B_1. Compute the mean and the covariance functions of Z.

Brownian Bridge

(2) We define the time reversal of Z by:

$$\forall t \in [0,1], \ Y_t = Z_{1-t}.$$

Show that both processes have the same law.

Exercise 8. A d-dimensional Brownian motion is a process of the form $(B_t = (B_t^1, \dots, B_t^d))_{t \geq 0}$, where $(B_t^i)_{t \geq 0}$, $1 \leq i \leq d$, are independent (real) Brownian motions. Show that for such a B and for a matrix U of size $d \times d$ with UU^* equal to the identity matrix, the process $(UB_t)_{t \geq 0}$ is also a d-dimensional Brownian motion.
(To simplify, you may choose $d = 2$.)

Exercise 9. Show that the probability that a Brownian motion is non-decreasing on a given interval $[a, b]$, $0 \leq a < b$, is zero.

Exercise 10. Let $(B_t^1)_{t \geq 0}$ and $(B_t^2)_{t \geq 0}$ be two independent Brownian motions. Show that $(B_t = 2^{-1/2}(B_t^1 + B_t^2))_{t \geq 0}$ is a Brownian motion.

Worksheet 2

Exercise 4. Let $(B_t)_{t\geq 0}$ be a B.M
Show that $(-B_t)_{t\geq 0}$ is also a B.M.

a) $-B_0 = -(B_0) = 0$.

b) $\operatorname{cov}(-B_s, -B_t) = E[(-B_s)(-B_t)] = \operatorname{cov}(B_s, B_t) = \inf(s,t)$

c) Is continuous $\mathbb{P}$ a.s

Exercise 5 B_t is B.M $(B_{t+a} - B_a)_{t\geq 0}$ $0 \leq t \leq a$

Show that $B_{a+t} - B_a$ is B.M and ind of B_t.

1) $W_0 = B_a - B_a = 0$

2) $\operatorname{cov}(W_s, W_t) = \inf(s+a, t+a) - a - a + a$

$= \inf(s,t) + a - a - a + a$.

Independence: Because it is Gaussian Process and $(B_t)_{0\leq t\leq a}$.

$\operatorname{cov}(W_t, B_t) = \operatorname{cov}(B_{t+a} - B_a, B_t)$

$= \operatorname{cov}(B_{t+a}, B_t) - \operatorname{cov}(B_a, B_t)$

$\inf(t+a, t) - \inf(a, t)$

$t - t = 0 \Rightarrow$ uncorrelated gaussian $\Rightarrow$ ind

6) $\tilde{B}_0 = 0 \qquad \tilde{B}_t = t\, B_{t^{-1}}$

i) $t \frac{1}{\sqrt{t}} B_1 = \sqrt{t}\, B_1 \sim N(0,t)$

$= B_t$. BM is Gaussian Process centered.

same law

ii) $\operatorname{cov}(\tilde{B}_t, \tilde{B}_s) = \operatorname{cov}\left(t B_{\frac{1}{t}}, s B_{\frac{1}{s}}\right) = \operatorname{cov}(B_t, B_s)$

$= \inf(t,s)$

$t B_{\frac{1}{t}}$ is G Process $\forall n \ \forall (t_1, t_2, t_n)$

$\sum_{i=1}^{n} \alpha_i t_i B_{\frac{1}{t_i}} = \sum_i \alpha_i B_{t_i}$ = Gaussian r.v.

Exercise 10

$B_t^1 \quad B_t^2$ ind B.M

$\Rightarrow B_t = 2^{-1/2}(B_t^1 + B_t^2)$ is B.M

1) $B_0 = 2^{-1/2}(B_0^1 + B_0^2) = 0$

2) $$\begin{aligned} \operatorname{cov}(B_t, B_s) &= \frac{1}{2}\operatorname{cov}(B_t^1 + B_t^2, B_s^1 + B_s^2) \\ &= \frac{1}{2} E\left[(B_t^1 + B_t^2)(B_s^1 + B_s^2)\right] \\ &= \frac{1}{2} E\left[B_t^1 B_s^1 + B_t^1 B_s^2 + B_t^2 B_s^1 + B_t^2 B_s^2\right] \\ &= \frac{1}{2}\left[\inf(t,s) + 0 + 0 + \inf(t,s)\right] \\ &= \inf(t,s) \end{aligned}$$

Example 1.4.14 BM

$$s \le t \qquad E[B_t \mid F_s] = E[(B_t - B_s) + B_s \mid F_s]$$

$$= E[B_t - B_s \mid F_s] + E[B_s \mid F_s]$$

$$\overset{\text{ind incr}}{=} E[B_t - B_s] + B_s$$

$$= E(B_t) - E[B_s] + B_s$$

$$= B_s.$$

Example 1.4.15 $(BM)^2$

$$B_t^2 - t$$

$$s < t \quad E[B_t^2 - t \mid F_s] = E[((B_t - B_s) + B_s)^2 - t \mid F_s]$$

$$= E[(B_t - B_s)^2 + 2(B_t - B_s)B_s + B_s^2 - t \mid F_s]$$

if $B_t - B_s \perp\!\!\!\perp \sigma$

also $(B_t - B_s)^2$

$t \perp\!\!\!\perp F_s$

$$\overset{\text{ind incr}}{=} E[(B_t - B_s)^2] + 2B_s E[B_t - B_s] + B_s^2 - t$$

$$= \underbrace{\cancel{\quad}}_{t-s} + B_s^2 - t. = B_s^2 - s.$$

Worksheet 2

Exercise 7 Brownian Bridge.

$(B_t)_{t \geq 0}$ a BM

$Z_t = B_t - t B_1 \qquad 0 \leq t \leq 1$

- $E[Z_t] = E[B_t] - t E[B_1] = 0 - 0 = 0$

- $\text{cov} = E[Z_t Z_s] = E[(B_t - tB_1)(B_s - sB_1)] = E[B_t B_s - s B_t B_1 - t B_1 B_s + ts B_1^2]$

$0 \leq s \leq t \leq 1$

$= \inf(t,s) - s \inf(t,1) - t \inf(1,s) + ts$

$= \inf(t,s) - st - ts + ts$

$\text{cov}(Z_t, Z_s) = \inf(t,s) - st.$

Show it is Gaussian Process

$(Z_t)_{t \geq 0}$ is GP iff $\forall (t_1, t_2, \dots t_n)$

$(Z_{t_1}, Z_{t_2}, \dots Z_{t_n})$

any $\alpha_1 Z_{t_1} + \alpha_2 Z_{t_2} + \dots$ is G r. variable.

$\alpha_1 (B_{t_1} - t_1 B_1) + \alpha_2 (B_{t_2} - t_2 B_1) + \dots$

$\underbrace{\sum \alpha_i B_{t_i}}_{\text{G r var}} - \underbrace{\sum \alpha_i t_i B_1}_{-\underbrace{\sum B_{t_i} B_1}_{\text{G r var.}}}$

$\text{cov}(B_t - tB_1, B_1) = t - t = 0 \Rightarrow$ ind for gaussians

Time reversal

part 2) $\quad Z_{1-t}$

$$E[Z_{1-t}] = E[B_{1-t} - (1-t)B_1] = E[B_{1-t}] - (1-t)E[B_1] = 0$$

$$\begin{aligned}
\operatorname{cov}(Z_{1-t}, Z_{1-s}) &= E[(B_{1-t} - (1-t)B_1)(B_{1-s} - (1-s)B_1)] \\
&= E[B_{1-t}B_{1-s}] + (1-t)(1-s)E[B_1^2] \\
&\quad - (1-t)E[B_1 B_{1-s}] - (1-s)E[B_{1-t}B_1] \\
&= \inf(1-t, 1-s) + (1-t)(1-s) \\
&\quad - (1-t)\inf(1, 1-s) - (1-s)\inf(1-t, 1) \\
&= \inf(1-t, 1-s) + (1-t)(1-s) - (1-t)(1-s) \\
&\qquad - (1-s)(1-t) \\
&= \inf(1-t, 1-s) - (1-s)(1-t)
\end{aligned}$$

WORKSHEET 3

In all the exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes the current probability space and $(B_t)_{t\geq 0}$ a (real) Brownian motion.

1. Conditional Expectation

Exercise 1.
Let $\mathcal{B}$ a σ-field of $\mathcal{A}$ and X be an independent r.v. of $\mathcal{B}$ of law $\mathcal{N}(0, \sigma^2)$. We consider a $\mathcal{B}$-measurable r.v. Y, and define the r.v. $Z = \exp(-\frac{\sigma^2}{2}Y^2 + XY)$.

(1) What is $\mathbb{E}[Z|\mathcal{B}]$?
(2) Show that $\mathbb{E}[Z] = 1$.

Exercise 2. Let X and Y be two independent r.v. of uniform law on $[0, 1]$. We set $U = \inf(X, Y)$ and $V = \sup(X, Y)$. What is $\mathbb{E}[U|V]$?

2. Discrete Martingales

Exercise 3. Let $(Y_n)_{n\geq 1}$ be a sequence of i.i.d.r.v. of law $\mathcal{U}([0, 1])$. We set $\mathcal{F}_n = \sigma(Y_1, ..., Y_n)$ and we define the process $(X_n)_{n\in\mathbb{N}}$ by $X_0 = 1$ and

$$X_{n+1} = \begin{cases} X_n + 1 & \text{if } 0 \leq Y_{n+1} < \frac{X_n}{X_n+1} \\ 0 & \text{if } \frac{X_n}{X_n+1} \leq Y_{n+1} \leq 1 \end{cases}$$

(1) Prove that X_n converges almost surely towards a r.v. X_∞.
(2) Prove that the process $(X_n)_{n\in\mathbb{N}}$ is an $(\mathcal{F}_n)$−martingale.
(3) Do we have $X_n = \mathbb{E}[X_\infty|\mathcal{F}_n]$? Is the martingale $(X_n)_{n\in\mathbb{N}}$ uniformly integrable?
(4) Let $T := \inf\{n \geq 0, X_n = 0\}$. Prove that T is an almost surely finite stopping time. Do we have the equality $\mathbb{E}[X_T] = \mathbb{E}[X_0]$?

Exercise 4. *Doob's decomposition.*
Prove that a submartingale (X_n) admits the following decomposition:

$$X_n = A_n + M_n, \qquad n \geq 0,$$

where(M_n) is a martingale and (A_n) is a predictable increasing process such that $A_0 = 0$.

Exercise 5. Let (X_n) be a submartingale and $a > 0$.

(1) Prove the *maximal inequality*:

$$a\,\mathbb{P}(\sup_{0\leq k\leq n} X_k \geq a) \leq \mathbb{E}[X_n \mathbb{1}_{\{\sup_{0\leq k\leq n} X_k \geq a\}}] \leq \mathbb{E}[X_n^+]$$

(2) Prove the following inequality:

$$a\,\mathbb{P}(\sup |X_k| \geq a) \leq 2\mathbb{E}[|X_n|] - \mathbb{E}[X_0] \leq 2\mathbb{E}[|X_n|] + \mathbb{E}[|X_0|]$$

Exercise 6. *Doob's inequality.*
Let (X_n) be a martingale.

(1) Prove that, for a non negative r.v. Z and $p \geq 1$,
$$\mathbb{E}[Z^p] = \int_0^\infty p a^{p-1} \mathbb{P}(Z \geq a) \mathrm{d}a$$

(2) We set $S_n := \sup_{0 \leq k \leq n} |X_k|$. Let $p > 1$ and $q = \frac{p}{p-1}$. Prove that $(|X_n|)$ is a submartingale and deduce the inequality:
$$\mathbb{E}\left[S_n^p\right] \leq q \, \mathbb{E}\left[|X_n| \, S_n^{p-1}\right]$$

(3) Prove Doob's inequality:
$$\|S_n\|_p \leq q \|X_n\|_p$$

Exercise 1 Worksheet ~~\$~~ 3

$X \sim N(0, \sigma^2)$

$X \perp\!\!\!\perp B \Rightarrow E[X|B] = E[X]$ or $\sigma(X) \not\subset B$

$B = \sigma(A)$

$Z = \exp\left(\frac{\sigma^2}{2} Y^2 + XY\right)$

$\sigma(Y) \subset B$ means Y is a "constant" $\Rightarrow E[Y|B] = Y$.

use Rule 7 $E[h(X,G)|F] = E[E_X[h(X,G)] | F]$

1) What is $E[Z/B]$

$$E[f(X,Y)|B] = \int_{\mathbb{R}} f(x,Y)\, d\mathbb{P}(x)$$ Because Y is constant. and

$$= E[f(X)]$$

$$\Rightarrow E[Z|B] = \int_{\mathbb{R}} e^{-\frac{\sigma^2 Y^2}{2} + xY} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}\, dx.$$

$$= \frac{e^{-\frac{\sigma^2 Y^2}{2}}}{\sqrt{2\pi}\,\sigma} \int_{\mathbb{R}} e^{xY - x^2/2\sigma^2}\, dx.$$

$$-a^2 + 2ab \pm b^2 = \frac{-x^2}{2\sigma^2} + xY + \dots$$

- $a = \frac{x}{\sigma\sqrt{2}}$
- $2ab = \frac{2x}{\sigma\sqrt{2}} b = xY \Rightarrow b = \frac{Y\sigma}{\sqrt{2}}$
- $b^2 = \frac{Y^2\sigma^2}{2}$

$$-a^2 + 2ab - b^2 = -\left(\frac{x}{\sigma\sqrt{2}}\right)^2 + xY - \left(\frac{Y\sigma}{\sqrt{2}}\right)^2 + \left(\frac{Y\sigma}{\sqrt{2}}\right)^2$$

$$= -\left(\frac{x}{\sigma\sqrt{2}} - \frac{Y\sigma}{\sqrt{2}}\right)^2 + \left(\frac{Y\sigma}{\sqrt{2}}\right)^2$$

$$= \frac{\cancel{e^{\frac{-\sigma^2 \Upsilon^2}{2}}}}{\sqrt{2\pi}\,\sigma} \int_{\mathbb{R}} e^{-\left(\frac{x}{\sigma\sqrt{2}} - \frac{\Upsilon\sigma}{\sqrt{2}}\right)^2 + \cancel{\left(\frac{\Upsilon\sigma}{\sqrt{2}}\right)^2}} dx.$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\mathbb{R}} e^{-\frac{1}{2}\left(\frac{x}{\sigma} - \Upsilon\sigma\right)^2} dx.$$

$$u = \frac{x}{\sigma} - \Upsilon\sigma \qquad du = \frac{dx}{\sigma}$$

$$= \frac{\cancel{\sigma}\,1}{\sqrt{2\pi}\,\cancel{\sigma}} \int_{\mathbb{R}} e^{-\frac{1}{2}u^2} du = 1$$

B) Rule 2

$$E[Z] = E[E[Z \mid B]]$$

$$= E[1] = 1.$$

we dont know the law of Υ $\iint_{\mathbb{R}^2} f(x,y)\, d\mathbb{P}_{XY}$

$$E[f(X,Y)] = E[E[f(X,Y) \mid F]]$$

E4
W3

Martingale M_n

$E[M_{n+1} | F_n] = M_n$

$E[|M_n|] < \infty$

$\sigma(M_n) \subset F_n$.

X_n SubMartingale (given)

$E[X_{n+1} | F_n] \geq X_n$

$E[|X_n|] < \infty$

$\sigma(X_n) \subset F_n$.

$X_n = M_n + A_n$

minkowski L^1

1) $\cancel{E|X_n| = E|M_n + A_n| \leq E|M_n| + E|A_n|}$

$E|A_n| = E|X_n - M_n| \overset{M,L}{\leq} \underbrace{E|M_n|}_{<\infty} + \underbrace{E|X_n|}_{<\infty} \overset{\Rightarrow}{<} \infty$

2) $\sigma(X_n) \subset F_n$.

$\sigma(M_n) \subset F_n$.

$\sigma(A_n) \subset F_{n-1}$ and $F_{n-1} \subset F_n$. $\Rightarrow$ $\sigma(A_n) \subset F_n$.

sum of measurable functions is also measurable.

A_n and M_n are adapted to F_n.

$\geq, + m$

3) $E[X_{n+1} | F_n] = E[M_{n+1} + A_{n+1} | F_n]$

$= E[M_{n+1} | F_n] + E[A_{n+1} | F_n]$

$= M_n + E[A_{n+1} | F_n]$

predictable.

$= M_n + A_{n+1}$

$= M_n + A_n + (A_{n+1} - A_n)$

$= X_n + (\text{pos.})$

$A_{n+1} - A_n > 0$
if A increasing

$A_0 = 0$

$\sigma(A_0) \subset F_{-1} = \phi$

no information

$\Rightarrow A_0 = \text{constant}$.

Doob's Decomposition

$\forall$ Discrete Stochastic Process (Adapted and Integrable)

$\exists!$ decomposition as $M_n + A_n$

Martingale + Predictable drift (integrable), $A_0 = 0$

if A_n increasing $\Rightarrow$ submartingale $X_n \leq E[X_{n+1} | F_n]$

if A_n decreasing $\Rightarrow$ supermartingale $X_n \geq E[X_{n+1} | F_n]$

Submartingale Ex: Coin Biased $p > \frac{1}{2}$

mart. $X_0 = E[X_t | F_0]$

$E[X_0] = E[E[X_t | F_0]]$

$$\boxed{E[X_0] = E[X_t]}$$

Exercise 5.

Markov Inequality

$$a\,P(X \geq a) \leq E[X] \qquad \text{if} \quad E[X] < \infty$$

$$\underbrace{a\,P(X \geq a)}_{a \int_a^\infty f(x)\,dx} = a\,E[\mathbb{1}_{X \geq a}] = E[a\,\mathbb{1}_{X \geq a}] \qquad \text{as} \quad x \geq a$$

$$\leq E[x\,\mathbb{1}_{X \geq a}] \leq E[x\,\mathbb{1}_{X \geq 0}]$$

$$= E[X]$$

$$a\,P(X \geq a) \leq E[X] \quad \|$$

Maximal Inequality $\quad \sup X_k \geq a = \bigcup_{k=0}^{n} E_k \qquad E_i \cap E_j = \emptyset$

$$a\,P(\sup X_k \geq a) \leq a\,E[\mathbb{1}_{\sup X_k \geq a}]$$

$$= a\,E[\mathbb{1}_{\bigcup^n E_k}] = a\,E\Big[\sum_{i=0}^{n} \mathbb{1}_{E_i}\Big]$$

$$= \sum_{i=0}^{n} E[a\,\mathbb{1}_{E_i}] \leq \sum_{i=0}^{n} E[X_i\,\mathbb{1}_{E_i}]$$

as $X_n \leq E[X_{n+k} | F_n]$ $\qquad \leq \sum_{i=0}^{n} E\big[E(X_m | F_i]\,\mathbb{1}_{E_i}\big]$

$E_i \in \sigma(X_0, X_i, \dots) \subset F_i$

$E_0 = \{X_0 \geq a\}$

$E_K = \{X_K \geq a \quad \forall i < K,\ X_i < a\}$

Worksheet 3

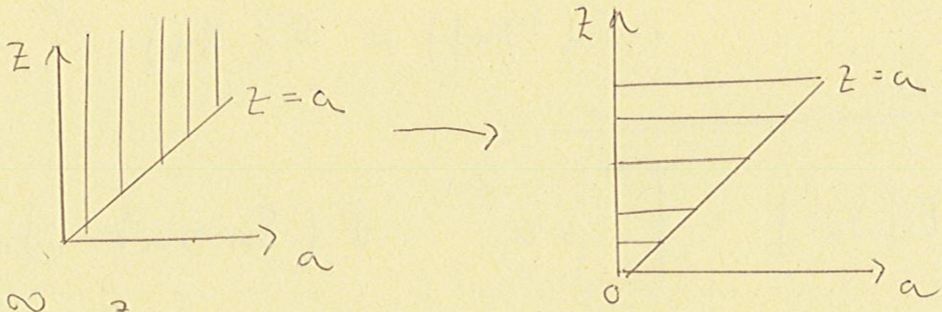
Exercise 6 Doob Inequality

1) $Z > 0$ $\quad p \geq 1$

$$E[Z^p] = \int_0^\infty z^p f(z)\,dz$$

$$E[Z^p] = \int_0^\infty p a^{p-1} \mathbb{P}(Z \geq a)\,da.$$

$$= \int_{a=0}^\infty p a^{p-1} \left(\int_{z=a}^\infty f(z)\,dz \right) da$$



$$= \int_{z=0}^\infty \int_{a=0}^{z} p a^{p-1} f(z)\,dz\,da.$$

$$= \int_{z=0}^\infty f(z) \left. a^p \right|_{a=0}^{a=z} dz$$

$$= \int_{z=0}^\infty z^p f(z)\,dz.$$

b) Prove $|X_n|$ is sub martingale.

X_n is Martingale

$S_n := \sup_{0 \le k \le n} |X_k|$

$p > 1$

$q = \frac{p}{p-1}$

$E[S_n^p] \le q \mathbb{E}[|X_n| S_n^{p-1}]$

- if X_n is Martingale $E[|X_n|] < \infty \Rightarrow E[|X_n|] < \infty$ ✓

$Y_n = |X_n| \overset{Mart}{=} |E[X_{n+1} | F_n]| \le E[|X_{n+1}| | F_n] = E[Y_{n+1} | F_n]$

- $Y_n \le E[Y_{n+1} | F_n]$ ✓

- $\sigma(|X_n|) \subset \sigma(X_n)$ ✓ Any $f(t, X_n)$ is adapted to $\sigma(X_n)$

$$E[S_n^p] = \int_0^\infty p a^{p-1} \mathbb{P}(S_n \geq a) \, da$$

$$= \int_0^\infty p a^{p-1} \underbrace{\mathbb{P}\left(\sup_{0 \le k \le n} |X_k| \geq a\right)}_{\text{Markov}} da$$

Markov $\le \frac{1}{a} E[\cancel{\quad}]$

Maximal Inequality. $\frac{1}{a} E[|X_k| \mathbb{1}_{\sup|X_k| \geq a}]$

$$\le \int_0^\infty p a^{p-2} E[|X_k| \mathbb{1}_{\sup|X_k| \geq a}] \, da$$

$$= \int_0^{\sup|X_k|} p a^{p-2} E[|X_k|] \, da$$

$$= E\left[|X_k| \int_0^{\sup|X_k|} p a^{p-2} \, da\right] = E\left[|X_k| \frac{p}{p-1} a^{p-1} \Big|_0^{S_n}\right]$$

$$= q E[|X_k| S_n^{p-1}].$$

Exercise 6 part c) previous $* \; E[S_n^p] \le q \, E[|X_n| S_n^{p-1}]$

Doob Inequality. $\|S_n\|_p \le q \|X_n\|_p$

$S_n := \sup_{0 \le k \le n} |X_k|$ $\quad X_k$ Martingale

$|X_k|$ sub Martingale

Hölder $E[XY] \le \|X\|_p \|Y\|_q$

$X = |X_n| \quad Y = S_n^{p-1}$

$\frac{1}{p} + \frac{1}{q} = 1$

$E[|X_n| S_n^{p-1}] \le \|X_n\|_p \|S_n^{p-1}\|_q$

$q + p = pq$

$\left(E((S_n^{p-1})^q)\right)^{1/q}$

$(p-1)q = pq - q$

$= q + p - q$

$= p.$

$\left(E \, S_n^p\right)^{1/q}$

$* \; E[S_n^p] \le q \|X_n\|_p \left(E[S_n^p]\right)^{1/q}$

$E[S_n^p]^{1-\frac{1}{q}} \le q \|X_n\|_p.$

$E[S_n^p]^{1/p} \le q \|X_n\|_p.$

$\|S_n\|_p \le q \|X_n\|_p$