Numerical methods MATHMODS course: Exercise sheet 2

1. Let $(X_t)_{t\geq 0}$ be the Ornstein-Uhlenbeck process given by

$$dX_t = \lambda(a - X_t)dt + \sigma dW_t, \quad t \geq 0, \quad X_0 = 1,$$

where as usual $(W_t)_{t\geq 0}$ is a standard 1-dimensional Brownian motion, and λ, a and σ are constants. Prove that

$$X_t \sim \mathcal{N}\left(a + (1-a)e^{-\lambda t}, \frac{\sigma^2}{2\lambda}(1 - e^{-2\lambda t})\right),$$

for every $t > 0$. Hence deduce that, if $\lambda > 0$, $X_t \to Z \sim \mathcal{N}(a, \sigma^2/2\lambda)$ in distribution as $t \to \infty$. What is the distribution of X_t if $X_0 \sim \mathcal{N}(a, \sigma^2/2\lambda)$? The limit measure is called the *invariant measure*.

Practical: Use the Euler scheme to simulate $N = 100,000$ paths of $(X_t)_{t\in[0,2]}$, with $X_0 = 1, a = 2, \lambda = 2$ and $\sigma = 0.3$. Plot approximations of the *density* $p_t(x)$, $x \in \mathbb{R}$ of X_t at times $t = 0.4$, 0.8, 1.2, 1.6 and 2.0 (note that at $t = 0$ both the actual and approximate densities are Dirac masses at 1). On each plot overlay the exact density function, and the density of the invariant measure calculated above. Observe the effect of changing N (for example take $N = 1000$) and λ.

Hint: In order to calculate the density of X_t at the 5 points in time, we only need to record the value of X_t at these particular points. Thus the first step might be to produce a raw data file that contains N rows (one for each simulated path) and five columns (one for each point in time), with the (i,j)-th entry being the position of particle i at the jth time step. This raw data file can then be used to create a histogram describing the density at each time step. For this you can either write your own procedure for creating histograms from raw data, or use a built-in function (for example numpy.histogram in Python).

2. This exercise is to prove the Lemma used in the proof of the L^p-convergence of the Euler scheme, which we recall. Suppose $(X_t)_{t\geq 0}$ is the $\mathbb{R}^n$-valued solution to the SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \qquad t \in [0, T],$$

where $(W_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued Brownian motion and $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ satisfy the usual conditions for existence and uniqueness of the solution: b and σ are globally Lipschitz uniformly in time and there exists a C_T such that

$$|b(t,x)| + |\sigma(t,x)| \leq C_T(1+|x|), \qquad \forall x \in \mathbb{R}^n,\ t \in [0,T].$$

b, σ grow at most linearly in x

Use this to show that for every $p \geq 1$ there exists a constant $C_T^{(p)}$ such that

(i)

$$\sup_{t\in[0,T]} \mathbb{E}\left[|X_t|^{2p}\right] \leq C_T^{(p)} \left(1+\mathbb{E}(|X_0|^{2p})\right);$$

(ii) for all $t, s \in [0,T]$ with $s \leq t$,

$$\mathbb{E}\left[|X_t - X_s|^{2p}\right] \leq C_T^{(p)} \left(1+\mathbb{E}(|X_0|^{2p})\right)(t-s)^p.$$

3. This exercise is designed to highlight the fact that the Euler scheme **does not always converge!** Reference: "Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients", Hutzenthaler, Jentzen and Kloeden, Proc. R. Soc. A 467 (2011).

Consider the stochastic differential equation in one dimension given by,

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \tag{1}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion, and μ and σ are functions $: \mathbb{R} \to \mathbb{R}$. Suppose that the initial condition is given by

$$X_0 = \xi,$$

where ξ is an almost surely finite real-valued random variable on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent of W_0. Suppose further that (1) has a unique solution on $[0,1]$, and that we would like to approximate this solution.

(i) Recall how to construct the Euler scheme approximation $(\bar{X}^N_{k/N})_{k=0,\dots,N}$ of the solution to (1) on $[0,1]$ with time step $\delta = \frac{1}{N}$.

Assume that the coefficients μ and σ are such that

$$\max\{|\mu(x)|, |\sigma(x)|\} \geq \frac{1}{C}|x|^\beta, \qquad \min\{|\mu(x)|, |\sigma(x)|\} \leq C|x|^\alpha,$$

for all $|x| \geq C$, and some constants $C \geq 1$ and $\beta > \alpha > 1$. In particular **we do not assume that the coefficients are globally Lipschitz**.

We will moreover suppose that $\mathbb{P}(\sigma(\xi) > 0) > 0$.

(ii) Use the assumption that $\mathbb{P}(\sigma(\xi) > 0) > 0$ to prove that there exists an integer $K \geq 1$ such that

$$\theta := \mathbb{P}\left[|\sigma(\xi)| \geq \frac{1}{K},\ |\xi| + |\mu(\xi)| \leq K\right] > 0.$$

Now define the constant

$$r_N := \max\left\{2, C, \left(2CN + 2C^2\right)^{\frac{1}{\beta-\alpha}}\right) \in [C, \infty),$$

and consider the sets

$$\Omega_N := \Big\{\omega \in \Omega \ :\ \left|W_{\frac{(k+1)}{N}}(\omega) - W_{\frac{k}{N}}(\omega)\right| \in \left[\frac{1}{N}, \frac{2}{N}\right] \ \forall k \in \{1, \ldots, N-1\},$$
$$\left|W_{\frac{1}{N}}(\omega) - W_0(\omega)\right| \geq K(r_N + K),$$
$$|\sigma(\xi)| \geq \frac{1}{K},\ |\xi| + |\mu(\xi)| \leq K\Big\},$$

for all $N \in \mathbb{N}$, where $K \geq 1$ is the fixed integer found in Part (ii).

(iii) Prove that
$$\frac{1}{CN} r_N^{\beta-\alpha} \geq \frac{1}{CN}\left(2CN + 2C^2\right) = 2 + \frac{2C}{N}.$$

(iv) Prove that for all N and $\omega \in \Omega_N$, it holds that

$$|\bar{X}^N_{1/N}(\omega)| \geq r_N.$$

(v) Prove by induction on k that for all N and $\omega \in \Omega_N$, it holds that

$$|\bar{X}^N_{k/N}(\omega)| \geq r_N^{\alpha^{k-1}}, \tag{2}$$

for all $k \in \{1, \ldots, N\}$. In particular, use this to show that

$$|\bar{X}^N_1(\omega)| \geq 2^{\alpha^{N-1}}, \tag{3}$$

for all $\omega \in \Omega_N$.

Hint: A good place to start is to note that under the induction hypothesis, it holds that
$$|\bar{X}^N_{k/N}| \geq r_N^{\alpha^{k-1}} \geq r_N \geq C \geq 1,$$

since $\alpha \geq 1$ and by definition of r_N. You may also like to use the elementary inequality $|a + b| \geq \max\{|a|, |b|\} - \min\{|a|, |b|\}$ for all $a, b \in \mathbb{R}$.

(vi) Use the Brownian scaling property to prove that, by the definition of Ω_N,

$$\mathbb{P}(\Omega_N) \geq \frac{\theta e^{-2}}{4} 2^{-N} N^{-N} \exp\left(-NK^2(r_N + K)^2\right),$$

for all $N \in \mathbb{N}$ (where θ is as in Part (i)). You may also use the inequalities

$$\mathbb{P}(|Z| \geq x) \geq \frac{xe^{-x^2}}{4}, \qquad \mathbb{P}(|Z| \in [x, 2x]) \geq \frac{xe^{-2x^2}}{2},$$

for any $x \geq 0$, where $Z \sim \mathcal{N}(0,1)$ is a standard normally distributed random variable.

(vii) Deduce that there exist constants $c_1, c_2 > 0$ and $\gamma > 1$ (independent of N) such that whenever N is large enough, it holds that

$$\mathbb{P}(\Omega_N) \geq c_1 \exp(-c_2 N^\gamma).$$

(viii) Use this result together with (3) to prove that

$$\lim_{N\to\infty} \mathbb{E}[|\bar{X}_1^N|] = \infty,$$

and thus that

$$\lim_{N\to\infty} \mathbb{E}[|\bar{X}_1^N|^p] = \infty,$$

for any $p \geq 1$.

(ix) Finally, returning the equation (1), suppose that the true solution is such that $\mathbb{E}[|X_1|^p] < \infty$ for some $p \in [1, \infty)$. Explain why Part (viii) shows that the Euler scheme fails to converge under the given conditions on μ and σ. Use the above results to give a simple example of an SDE with continuous coefficients such that $\mathbb{E}[|X_1|] < \infty$, but for which the Euler scheme does not converge on the interval $[0, 1]$.

Homework 2

Exercise 1

$$dX_t = \lambda(a - X_t)\,dt + \sigma\, dB_t$$

$$\hat{X}_t' = a - X_t \qquad d\hat{X}_t = -dX_t$$

$$-d\tilde{X}_t = +\lambda \hat{X}_t\, dt + \sigma\, dB_t.$$

Integrating factor

$$I\,d\tilde{X}_t + \lambda I\, d\tilde{X}_t = -I\sigma\, dB_t. \qquad I_t = e^{+\int_0^t \lambda\, dt}$$

$$\frac{d}{dt} I\hat{X}_t = -I\sigma\, dB_t$$

$$I_t \hat{X}_t - I_0 \hat{X}_0 = -\int_0^t I_t \sigma\, dB_t$$

$$I_t \tilde{X}_t = I_0 \hat{X}_0 + \int_0^t e^{+\int_0^t \lambda dt} \sigma\, dB_t$$

$$\hat{X}_t = \tilde{X}_0 I_t^{-1} + \Big(\int_0^t \underbrace{e^{+\int_0^t \lambda dt}}_{e^{+\lambda t}} \sigma\, dB_t.\Big) I_t^{-1}$$

$$\tilde{X}_t = (a - X_0)e^{-\lambda t} + \Big(\int_0^t e^{+\lambda t} \sigma\, dB_t\Big) e^{-\lambda t}.$$

$$a - X_t = (a - X_0)e^{-\lambda t} + e^{-\lambda t}\int_0^t e^{+\lambda t}\sigma\, dB_t$$

$$X_t = a + (X_0 - a)e^{-\lambda t} + \underbrace{e^{-\lambda t}\int_0^t e^{\lambda t} \sigma\, dB_t.}_{I\Gamma\hat{o} \quad EI = 0}$$

- $$\mathbb{E} X_t = a + (X_0 - a)e^{-\lambda t}$$

$$\operatorname{var} X_t = \operatorname{var}\left(a+(X_0-a)e^{-\lambda t}\right) + \operatorname{var}\left(e^{-\lambda t}\int_0^t e^{\lambda t}\sigma\, dB_t\right)$$

$$= 0 + \sigma^2 e^{-2\lambda t}\operatorname{var}\left(\int_0^t e^{\lambda s}\, dB_s\right)$$

$$\operatorname{var}\int_0^t e^{\lambda s}\, dB_s = \mathbb{E}\left(\int_0^t e^{\lambda s}\, dB_s\right)^2 - 0$$

$$= \int_0^t e^{2\lambda s}\, ds = \frac{e^{2\lambda s}}{2\lambda}\Big|_0^t$$

$$\operatorname{var} X_t = \sigma^2 e^{-2\lambda t}\left[\frac{e^{2\lambda t}-1}{2\lambda}\right] = \frac{\sigma^2}{2\lambda}\left[1-e^{-2\lambda t}\right]$$

2) $X_t = X_0 + \int_0^t b\,ds + \int_0^t \sigma\,dB_s$

$$|a+b|^{2p} \le C^{(p)}\left(|a|^{2p} + |b|^{2p}\right)$$

$$|X_t|^{2p} = \left|X_0 + \int_0^t b\,ds + \int_0^t \sigma\,dB_s\right|^{2p}$$

$$\le C_1^{(p)}\left(\left|X_0 + \int_0^t b\,ds\right|^{2p} + \left|\int_0^t \sigma\,dB_s\right|^{2p}\right)$$

$$\le C_1^{(p)}\left(C_2^{(p)}\left(|X_0|^{2p} + \left|\int_0^t b\,ds\right|^{2p}\right) + \left|\int_0^t \sigma\,dB_s\right|^{2p}\right)$$

$$\le C^{(p)}\left(|X_0|^{2p} + \underbrace{\left|\int_0^t b\,ds\right|^{2p}}_{I} + \underbrace{\left|\int_0^t \sigma\,dB_s\right|^{2p}}_{II}\right)$$

I) $$\left|\int_0^t b\,ds\right| \overset{\text{Hölder}}{\le} \left(\int_0^t b^{2p}\,ds\right)^{1/2p} \underbrace{\left(\int_0^t 1^{2p/2p-1}\right)^{\frac{2p-1}{2p}}}_{T^{2p-1/2p}} \qquad \frac{1}{2p} + \frac{1}{q} = 1 \qquad \frac{1}{q} = 1 - \frac{1}{2p}$$

II) $$\mathbb{E}\sup_u \left|\int_0^u \sigma\,dB_s\right|^{2p} \le K_m\, \mathbb{E}\left(\int_0^t \sigma^2\,ds\right)^p \quad \text{(BDG)}$$

now Hölder $$\int_0^t \sigma^2\,ds \le \left(\int_0^t \sigma^{2p}\,ds\right)^{1/p}\left(\int_0^t 1^{2p/p-1}\,ds\right)^{\frac{p-1}{p}} \qquad \frac{1}{p} + \frac{1}{q} = 1$$

$$\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$$

$$\Rightarrow \mathbb{E}\sup |X_t|^{2p} = C^{(p)}\,\mathbb{E}|X_0|^{2p} + \mathbb{E}\int_0^t b^{2p}\,ds + \underbrace{\mathbb{E}\sup_u \int_0^u \sigma\,dB_s}$$

$$+\,\mathbb{E}\int_0^t \sigma^{2p}\,ds$$

$$|X_t|^{2p} \le C_T^{(p)} \left(|X_0|^{2p} + \int_0^t b^{2p} ds + \left| \int_0^t \sigma \, dB_s \right|^{2p} \right)$$

$$\left(\int_0^t \sigma \, dB_s \right)^{2p} \le \sup_u \left(\int_0^u \sigma \, dB_s \right)^{2p} \Rightarrow \mathbb{E} \left(\int_0^t \sigma \, dB_s \right)^{2p} \le \mathbb{E} \sup_u \int_0^u \cdots$$

use BDG

☆

Example

$$|b^{2p}(t) x^{2p}| + |\sigma^{2p}(t) x^{2p}| \le |x^{2p}| \overbrace{(b^{2p} + \sigma^{2p})}^{+}$$

$$\le (|x^{2p}| + 1) \underbrace{(b^{2p} + \sigma^{2p})}_{K}$$

$$\Rightarrow \mathbb{E} |X_t|^{2p} \le C_T^{(p)} \left(\mathbb{E} X_0^{2p} + \mathbb{E} \int_0^t \underbrace{b^{2p} + \sigma^{2p}}_{\le K (1 + |X_t|^{2p})} ds \right)$$

$$\underbrace{\sup \mathbb{E} |X_t|^{2p}}_{y} \le C_T^{(p)} \left(\mathbb{E} X_0^{2p} + Kt + \underbrace{\sup_w \mathbb{E} \int_0^w |X_w|^{2p} d_w}_{\le \int_0^T \underbrace{\sup_t \mathbb{E} |X_t|^{2p}}_{\int_0 y} ds} \right)$$

growall Inequality

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for every $t > 0$. Hence deduce that, if $\lambda > 0$, $X_t \to Z \sim \mathcal{N}(a, \sigma^2/2\lambda)$ in distribution as $t \to \infty$. What is the distribution of X_t if $X_0 \sim \mathcal{N}(a, \sigma^2/2\lambda)$? The limit measure is called the *invariant measure*.

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Hint: In order to calculate the density of X_t at the 5 points in time, we only need to record the value of X_t at these particular points. Thus the first step might be to produce a raw data file that contains N rows (one for each simulated path) and five columns (one for each point in time), with the (i, j)-th entry being the position of particle i at the jth time step. This raw data file can then be used to create a histogram describing the density at each time step. For this you can either write your own procedure for creating histograms from raw data, or use a built-in function (for example numpy.histogram in Python).

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$$|b(t,x)| + |\sigma(t,x)| \leq C_T(1 + |x|), \qquad \forall x \in \mathbb{R}^n,\ t \in [0,T].$$

Use this to show that for every $p \geq 1$ there exists a constant $C_T^{(p)}$ such that

(i)

$$\sup_{t\in[0,T]} \mathbb{E}\left[|X_t|^{2p}\right] \leq C_T^{(p)}\left(1 + \mathbb{E}(|X_0|^{2p})\right);$$

(ii) for all $t, s \in [0,T]$ with $s \leq t$,

$$\mathbb{E}\left[|X_t - X_s|^{2p}\right] \leq C_T^{(p)}\left(1 + \mathbb{E}(|X_0|^{2p})\right)(t-s)^p.$$

3. This exercise is designed to highlight the fact that the Euler scheme **does not always converge!** Reference: "Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients", Hutzenthaler, Jentzen and Kloeden, Proc. R. Soc. A 467 (2011).

Consider the stochastic differential equation in one dimension given by,

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where $(W_t)_{t\geq 0}$ is a standard Brownian motion, and μ and σ are functions : $\mathbb{R} \to \mathbb{R}$. Suppose that the initial condition is given by

$$X_0 = \xi,$$

where ξ is an almost surely finite real-valued random variable on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent of W_0. Suppose further that (1) has a unique solution on $[0,1]$, and that we would like to approximate this solution.

(i) Recall how to construct the Euler scheme approximation $(X^N_{k/N})_{k=0,\dots,N}$ of the solution to (1) on $[0,1]$ with time step $\delta = \frac{1}{N}$.

Assume that the coefficients μ and σ are such that

$$\max\{|\mu(x)|, |\sigma(x)|\} \geq \frac{1}{C}|x|^\beta, \qquad \min\{|\mu(x)|, |\sigma(x)|\} \leq C|x|^\alpha,$$

for all $|x| \geq C$, and some constants $C \geq 1$ and $\beta > \alpha > 1$. In particular **we do not assume that the coefficients are globally Lipschitz.**

We will moreover suppose that $\mathbb{P}(\sigma(\xi) > 0) > 0$.

(ii) Use the assumption that $\mathbb{P}(\sigma(\xi) > 0) > 0$ to prove that there exists an integer $K \geq 1$ such that

$$\theta := \mathbb{P}\left[|\sigma(\xi)| \geq \frac{1}{K},\ |\xi| + |\mu(\xi)| \leq K\right] > 0.$$

Proof. Suppose otherwise, so that $\forall K \geq 1$, $\mathbb{P}\left[|\sigma(\xi)| \geq \frac{1}{K},\ |\xi| + |\mu(\xi)| \leq K\right] = 0$. Now it is clear that

$$\{|\sigma(\xi)| > 0\} = \{|\sigma(\xi)| > 0,\ |\xi| + |\mu(\xi)| < \infty\} \subset \bigcup_{K \geq 1} \left\{|\sigma(\xi)| \geq \frac{1}{K},\ |\xi| + |\mu(\xi)| \leq K\right\}.$$

Indeed, if ω is not in the RHS, then it is not in the LHS. Therefore

$$\mathbb{P}(\{|\sigma(\xi)| > 0\}) \leq \bigcup_{K \geq 1} \mathbb{P}\left(\left\{|\sigma(\xi)| \geq \frac{1}{K},\ |\xi| + |\mu(\xi)| \leq K\right\}\right) = 0,$$

which is a contradiction. □

Now define the constant

$$r_N := \max\left\{2, C, (2CN + 2C^2)^{\frac{1}{\beta-\alpha}}\right) \in [C, \infty),$$

and consider the sets

$$\Omega_N := \left\{\omega \in \Omega \ :\ \left|W_{\frac{(k+1)}{N}}(\omega) - W_{\frac{k}{N}}(\omega)\right| \in \left[\frac{1}{N}, \frac{2}{N}\right]\ \forall k \in \{1, \dots, N-1\},\right.$$
$$\left|W_{\frac{1}{N}}(\omega) - W_0(\omega)\right| \geq K(r_N + K),$$
$$\left.|\sigma(\xi)| \geq \frac{1}{K},\ |\xi| + |\mu(\xi)| \leq K\right\},$$

for all $N \in \mathbb{N}$, where $K \geq 1$ is the fixed integer found in Part (ii).

(iii) Prove that

$$\frac{1}{CN} r_N^{\beta-\alpha} \geq \frac{1}{CN}\left(2CN + 2C^2\right) = 2 + \frac{2C}{N}.$$

Proof. Use fact that $x \mapsto x^{\beta-\alpha}$ is increasing since $\beta - \alpha > 0$. □

(iv) Prove that for all N and $\omega \in \Omega_N$, it holds that

$$|X^N_{1/N}(\omega)| \geq r_N.$$

Proof. We have

$$\begin{aligned}
|X^N_{1/N}(\omega)| &= \left|\xi(\omega) + \frac{1}{N}\mu(\xi(\omega)) + \sigma(\xi(\omega))\left(W_{\frac{1}{N}}(\omega) - W_0(\omega)\right)\right| \\
&\geq |\sigma(\xi(\omega))|\left|W_{\frac{1}{N}}(\omega) - W_0(\omega)\right| - |\xi(\omega)| - \frac{1}{N}|\mu(\xi(\omega))| \\
&\geq |\sigma(\xi(\omega))|\left|W_{\frac{1}{N}}(\omega) - W_0(\omega)\right| - K \\
&\geq (r_N + K) - K = r_N,
\end{aligned}$$

by definition of Ω_N. □

(v) Prove by induction on k that for all N and $\omega \in \Omega_N$, it holds that

$$|X^N_{k/N}(\omega)| \geq r_N^{\alpha^{k-1}}, \tag{2}$$

for all $k \in \{1, \dots, N\}$. In particular, use this to show that

$$|X^N_1(\omega)| \geq 2^{\alpha^{N-1}}, \tag{3}$$

for all $\omega \in \Omega_N$.
Hint: A good place to start is to note that under the induction hypothesis, it holds that

$$|X^N_{k/N}| \geq r_N^{\alpha^{k-1}} \geq r_N \geq C \geq 1,$$

since $\alpha \geq 1$ and by definition of r_N. You may also like to use the elementary inequality $|a + b| \geq \max\{|a|, |b|\} - \min\{|a|, |b|\}$ for all $a, b \in \mathbb{R}$.

Proof. Assume it holds for some $k \in \{1, \dots, N\}$. Then in particular

$$|X^N_{k/N}| \geq r_N^{\alpha^{k-1}} \geq r_N \geq C \geq 1, \tag{4}$$

since $\alpha \geq 1$. Then

$$\begin{aligned}
|X^N_{(k+1)/N}(\omega)| &= \left|X^N_{k/N}(\omega) + \frac{1}{N}\mu(X^N_{k/N}(\omega)) + \sigma(X^N_{k/N}(\omega))\left(W_{\frac{(k+1)}{N}}(\omega) - W_{\frac{k}{N}}(\omega)\right)\right| \\
&\geq \left|\frac{1}{N}\mu(X^N_{k/N}(\omega)) + \sigma(X^N_{k/N}(\omega))\left(W_{\frac{(k+1)}{N}}(\omega) - W_{\frac{k}{N}}(\omega)\right)\right| - |X^N_{k/N}(\omega)| \\
&\geq \max\left\{\frac{1}{N}|\mu(X^N_{k/N}(\omega))|,\ |\sigma(X^N_{k/N}(\omega))|\left|W_{\frac{(k+1)}{N}}(\omega) - W_{\frac{k}{N}}(\omega)\right|\right\} \\
&\quad - \min\left\{\frac{1}{N}|\mu(X^N_{k/N}(\omega))|,\ |\sigma(X^N_{k/N}(\omega))|\left|W_{\frac{(k+1)}{N}}(\omega) - W_{\frac{k}{N}}(\omega)\right|\right\} - |X^N_{k/N}(\omega)| \\
&\geq \frac{1}{N}\max\left\{|\mu(X^N_{k/N}(\omega))|,\ |\sigma(X^N_{k/N}(\omega))|\right\} \\
&\quad - \frac{2}{N}\min\left\{|\mu(X^N_{k/N}(\omega))|,\ |\sigma(X^N_{k/N}(\omega))|\right\} - |X^N_{k/N}(\omega)|.
\end{aligned}$$

as required. ∎

(viii) Use this result together with (3) to prove that

$$\lim_{N\to\infty} \mathbb{E}[|\bar{X}_1^N|] = \infty,$$

and thus that

$$\lim_{N\to\infty} \mathbb{E}[|\bar{X}_1^N|^p] = \infty,$$

for any $p \geq 1$.

Proof. We have that

$$\begin{aligned}\lim_{N\to\infty} \mathbb{E}[|\bar{X}_1^N|] &\geq \lim_{N\to\infty} \mathbb{E}[\mathbf{1}_{\Omega_N}|\bar{X}_1^N|] \\ &\geq \lim_{N\to\infty} 2^{\alpha^{N-1}}\mathbb{P}(\Omega_N) \\ &\geq c_1 \lim_{N\to\infty} 2^{\alpha^{N-1}} e^{-c_2 N^{\gamma}} = \infty,\end{aligned}$$

since α^{N-1} grows faster than N^{γ}! ∎

(ix) Finally, returning the equation (1), suppose that the true solution is such that $\mathbb{E}[|X_1|^p] < \infty$ for some $p \in [1, \infty)$. Explain why Part (viii) shows that the Euler scheme fails to converge under the given conditions on μ and σ. Use the above results to give a simple example of an SDE with continuous coefficients such that $\mathbb{E}[|X_1|] < \infty$, but for which the Euler scheme does not converge on the interval $[0, 1]$.

Proof. For example, take

$$dX_t = -X_t^3 dt + dW_t.$$

Then $\mu(x) = -x^3$, $\sigma = 1$ and for all $|x| \geq 1$,

$$\max\{|\mu(x)|, |\sigma(x)|\} = \max\{|x^3|, 1\} = |x|^3$$

and

$$\min\{|\mu(x)|, |\sigma(x)|\} = \min\{|x^3|, 1\} = 1 \leq |x|^2,$$

so can take $C = 1$ $\beta = 3$ and $\alpha = 2$, for example. ∎

x^3 is not globally Liptschitz but Locally