Numerical methods MATHMODS course: Exercise sheet 3

1. Let $(B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion starting from 0. Let $a > 0$ and define

$$\tau_a := \inf\{t > 0 : B_t \geq a\}$$

(i) Prove that $\mathbb{P}(\tau_a < \infty) = 1$.
Hint: Consider the stochastic process $(Z_t)_{t\geq 0}$ defined by

$$Z_t = \exp\left(\beta B_t - \frac{\beta^2}{2}t\right), \quad t \geq 0, \ \beta > 0.$$

Use Novikov's theorem to show that Z_t is a martingale, and thus that

$$\mathbb{E}\left(Z_{t\wedge\tau_a}\right) = 1, \quad \forall t \geq 0.$$

Write

$$\mathbb{E}\left(Z_{t\wedge\tau_a}\right) = \mathbb{E}\left(Z_{t\wedge\tau_a} 1_{\{\tau_a=\infty\}}\right) + \mathbb{E}\left(Z_{t\wedge\tau_a} 1_{\{\tau_a<\infty\}}\right)$$

and use this to show that

$$1 = \mathbb{E}\left(\exp\left(\beta a - \frac{\beta^2}{2}\tau_a\right) 1_{\{\tau_a<\infty\}}\right). \tag{1}$$

Finally, take the limit in the above as $\beta \to 0$.

(ii) Use part (i) and (1) to prove that the Laplace transform of τ_a is given by

$$\mathbb{E}(e^{-\lambda\tau_a}) = e^{-a\sqrt{2\lambda}}, \qquad \forall \lambda > 0.$$

2. Practical: Let $(W_t)_{t\geq 0}$ be a standard Brownian motion starting from 0 and define the first hitting time of 1 by

$$\tau_1 = \inf\{t > 0 : W_t \geq 1\}.$$

Write code that approximates τ_1 using the first naive method described in lectures. Use this code to produce a figure of the approximate density of τ_1 on $[0, 1]$ (using for example 100000 particles and time steps of 0.001). Overlay on this figure the exact density calculated from lectures, which we recall to be

$$\mathbb{P}(\tau_1 \in dt) = \frac{1}{\sqrt{2\pi}} t^{-\frac{3}{2}} e^{-\frac{1}{2t}}, \qquad t \geq 0.$$

Modify your code to use one of the other two methods from lectures, and compare the results.

3. *Brownian bridge.*

A zero-mean continuous Gaussian process $B = (B(t))_{t\in[0,1]}$ is called a Brownian bridge if the covariance function of B is given by

$$C(s,t) = \min\{s,t\} - st, \qquad \forall s,t \in [0,1].$$

Show that $B(t) = W(t) - tW(1) = (1-t)W\left(\frac{t}{1-t}\right)$, $t \in [0,1]$, where $(W(t))_{t\in[0,1]}$ is a standard Brownian motion.

Let $(W(t))_{t\geq 0}$ be a 1-dimensional Brownian motion, and fix three deterministic times $T_0 < T_1 < T_2$. Define

$+\alpha W(T_1) - \alpha W(T_1)$ $\quad$ $W(T_1) - W(T_0)$

$$Y_{\alpha,\beta} = W(T_1) - \alpha W(T_0) - \beta W(T_2), \qquad \alpha, \beta \geq 0.$$

(a) What is the law of $Y_{\alpha,\beta}$?

(b) Find conditions on α and β such that: (i) $Y_{\alpha,\beta}$ and $W(T_0)$ are independent; (ii) $Y_{\alpha,\beta}$ and $W(T_2)$ are independent; and (iii) $Y_{\alpha,\beta}$ is independent of both $W(T_0)$ and $W(T_2)$.

(c) Write $W(T_1)$ as a function of $W(T_0)$, $W(T_2)$ and $Y_{\alpha,\beta}$. Assume that you have already simulated $W(T_0), W(T_2), \ldots, W(T_{2N})$ for some $N \in \mathbb{N}$, and you want to know the path on a finer grid $T_0 < T_1 < T_2 < \cdots < T_{2N}$. Describe an algorithm to generate $W(T_1), W(T_3), \ldots, W(T_{2N-1})$.

Homework 3

Exercise 1 $\quad \mathbb{P}(T_a < \infty) = 1$

Novikov Thm: let X_t real valued, adapted in $(\Omega, \mathcal{F}_t, \mathbb{P})$ and B_t a BM

for X_t, $0 \le t \le T$

if $\mathbb{E}\left[e^{\frac{1}{2}\int_0^T |X_t|^2 dt} \right] < \infty \Rightarrow Z_t = e^{\int_0^t X_s dB_s - \frac{1}{2}\int_0^t X_s^2 ds}$

(BIG T) is a Martingale in $\mathbb{P}$. (small t) $0 \le t \le$

We have $X_t = \beta$ so $\mathbb{E}\left[e^{\frac{1}{2}\beta^2 t} \right] < \infty$

and $Z_t = \exp\left[\beta B_t - \frac{\beta^2}{2} t \right]$ is a Martingale

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Proof

$s \le t$

$$\mathbb{E}\left[e^{\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds} \Big| F_s \right] = E[Z_t | F_s]$$

$$= e^{-\frac{1}{2}\int_0^t \theta_s^2 ds} \underbrace{\mathbb{E}\left[e^{\int_0^t \theta_s dB_s} \Big| F_s \right]}_{\int_0^s + \int_s^t}$$

$\int_0^s | F_s \Rightarrow$ cte

$\int_s^t \perp\!\!\!\perp F_s$ ind incr of BM

$$= e^{-\frac{1}{2}\int_0^t \theta_s^2 ds} e^{\int_0^s \theta_s dB_s} \underbrace{\mathbb{E}\left[e^{\int_s^t \theta_s dB_s} \right]}_{\text{momentum generator}}$$

$\int_s^t \theta_s dB_s \sim N(0, \int_s^t \theta_s^2 ds$

$e^{m\lambda + \frac{1}{2}\lambda^2 \text{var}}$

$$= e^{-\frac{1}{2}\int_0^t \theta_s^2 ds + \int_0^s \theta_s dB_s + \frac{1}{2}\int_s^t \theta_s^2 ds}$$

$$= e^{-\frac{1}{2}\int_0^s \theta_s^2 ds + \int_0^s \theta_s dB_s} = Z_s \quad \checkmark$$

Doob's Optional Stopping Theorem

Let Z_t a Martingale, T_a an stopping Time.

$$\mathbb{E}[Z_{T_a}] = \mathbb{E}[Z_0]$$

$\mathbb{E} Z_0 = 1$ $\qquad t \wedge T_a$ is stopping time.

$$\mathbb{E}[Z_{t\wedge T_a}] = \mathbb{E}[Z_{t\wedge T_a} \mathbb{1}_{T_a = \infty}] + \mathbb{E}[Z_{t\wedge T_a} \mathbb{1}_{T_a < \infty}]$$

$$1 = \underbrace{\mathbb{E}[Z_{t\wedge T_a} \mathbb{1}_{T_a=\infty}]}_{=0 \text{ as } t\to\infty} + \mathbb{E}[Z_{t\wedge T_a} \mathbb{1}_{T_a<\infty}]$$

$t \to \infty \quad Z_{T_a}$

$$1 = \mathbb{E}\left[e^{\beta B_{T_a} - \frac{\beta^2 T_a}{2}} \mathbb{1}_{T_a < \infty}\right]$$

when $\beta \to 0$

$$1 = \mathbb{E}[\mathbb{1}_{T_a<\infty}]$$

$$\underline{1 = \mathbb{P}(T_a < \infty)}$$

$$1 = \mathbb{E}\left(e^{\beta a - \frac{\beta^2}{2} T_a} \mathbb{1}_{T_a < \infty}\right) \Rightarrow e^{-\beta a} = \mathbb{E}\left[e^{-\frac{\beta^2}{2} T_a}\right]$$

$\lambda = \frac{\beta^2}{2} \quad \beta = \sqrt{2\lambda}$

$$\Rightarrow e^{-\sqrt{2\lambda}\, a} = \mathbb{E}[e^{-\lambda T_a}]$$

laplace transform of T_a

Exercise 3

$$Y_{\alpha,\beta} = -\beta B_{T_2} + B_{T_1} - \alpha B_{T_0}$$

$$= -\beta B_{T_2} + \underline{\beta B_{T_1} - \beta B_{T_1}} + B_{T_1} - \alpha B_{T_0}$$

$$= -\beta (B_{T_2} - B_{T_1}) - (\beta - 1) B_{T_1} - \alpha B_{T_0}$$

$$-(\beta-1) B_{T_1} + \underline{(\beta-1) B_{T_0} - (\beta-1) B_{T_0}} - \alpha B_{T_0}$$

$$= -\beta (B_{T_2} - B_{T_1}) - (\beta-1) (B_{T_1} - B_{T_0}) - (\beta - 1 + \alpha) B_{T_0}$$

$\operatorname{var} Y_{\alpha\beta} =$ or use $\operatorname{var} X+Y = \operatorname{var} X + \operatorname{var} Y + 2 \operatorname{cov} XY$.

$$\beta^2 (T_2 - T_1) + (\beta-1)^2 (T_1 - T_0) + (\beta + \alpha - 1)^2 T_0$$

$$\operatorname{cov} (B_{T_1}, B_{T_0}) = \mathbb{E} [B_{T_1} B_{T_0}] = \mathbb{E} [(B_{T_1} - B_{T_0} + B_{T_0}) B_{T_0}]$$

$$= \mathbb{E} [(B_{T_1} - B_{T_0}) B_{T_0}] + \mathbb{E} (B_{T_0})^2$$

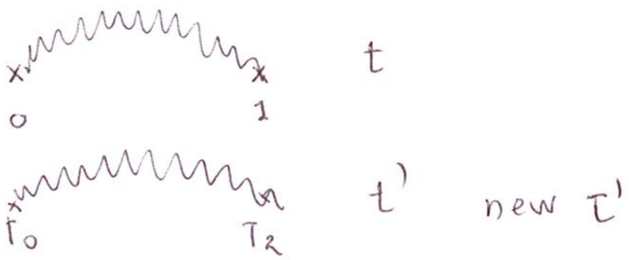
$$= T_0 .$$

$$\operatorname{cov} (B_{T_2}, B_{T_0}) = T_0 .$$

$$\operatorname{cov} (Y_{\alpha,\beta}, B_{T_2}) = T_1 - \alpha T_0 - \beta T_2$$

$$\operatorname{cov} (Y_{\alpha,\beta}, B_{T_0}) = T_0 (1 - \alpha - \beta)$$

simulate a Brownian Bridge between T_0 and T_2



$$T_0 + (T_2 - T_0)\,t = t'$$

so $$t = \frac{t' - T_0}{T_2 - T_0}.$$

substitute in B Bridge.

$$\cancel{(1-t)\,B\left(\frac{t}{1-t}\right)}$$

Simulate Gaussian with cov = Bridge
substitute t by t'

check

$$B_t - t B_1 \overset{?}{\overset{L}{=}} (1-t)\,B\left(\frac{t}{1-t}\right) = (1-t)\frac{\sqrt{t}}{\sqrt{1-t}}\,B_1 = \sqrt{t}\,(\sqrt{1-t})\,B_1$$

$\shortparallel$

$\sqrt{t}\,B_1 - t B_1$

$(\sqrt{t} - t)\,B_1$

$\sqrt{t}\,(1 - \sqrt{t})\,B_1$

Numerical methods MATHMODS course: Exercise sheet 4

1. Suppose that there exists a risky asset with stochastic price process $(S_t)_{t\geq 0}$ and a risk-free asset $(B_t)_{t\geq 0}$ that evolves at the risk-free rate r i.e. $dB_t = rB_t dt$. Let $\Pi(t, S_t)$ be a portfolio built on this market (i.e. it consists of long and short positions in both the risky and risk-free asset), and suppose that the evolution of Π is deterministic.

Definition: There exists **arbitrage** in the market if there is the possibility of making a risk free profit.

Under the assumption that there is no arbitrage in the market, prove that Π must evolve at the risk-free rate r.

2. Suppose that we in the Black-Scholes set up: we have a risky asset modeled as a geometric Brownian motion (with constant drift μ and volatility σ), and a risk-free asset that evolves deterministically according to the interest rate r.

Let $V(t, S)$, $t \geq 0$, $S \in \mathbb{R}$ be the price of an option on the asset at time t and when the price of the underlying is S. Suppose the option has expiry T and payoff function given by $\Psi : \mathbb{R} \to \mathbb{R}$. Recall the general pricing formula for V is

$$V(t, S) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left(\Psi(S_T)|S_t = S\right),$$

where (under the risk-neutral measure $\mathbb{Q}$),

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

and $(W_t)_{t\geq 0}$ is a standard 1-dimensional Brownian motion. Show that this implies

$$V(t, S) = e^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi\left(Se^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}z}\right)e^{-\frac{z^2}{2}}dz.$$

In the case of a European call option with strike price K, so that $\Psi(S) = \max\{S_T - K, 0\}$, show that in fact we can explicitly write

$$V(t, S) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \tag{1}$$

where

$$d_i = \frac{\log\left(\frac{S}{K}\right) + \left(r + (-1)^{i+1}\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad i \in \{1,2\},$$

and $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}dz$ is the cumulative distribution function for a standard $\mathcal{N}(0,1)$ random variable. This is the world-famous **Black-Scholes formula** for the price of a European call option!

3. Generalised Feynman-Kac: Suppose that there exists a solution $F \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R})$ to the equation

$$\begin{cases} \partial_t F(t,x) + \mathcal{L}F(t,x) - k(t,x)F(t,x) + g(t,x) = 0, & t \in [0,T),\ x \in \mathbb{R}^d \\ F(T,x) = \Psi(x) \end{cases},$$

where

- $$\mathcal{L} = \sum_{i=1}^{d} b_i(\cdot)\partial_i + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(\cdot)\partial_{ij},$$

  and the $d \times d$ matrix $a(x) = (a_{ij}(x))$ can be written as $\sigma\sigma^T(x)$ for every $x \in \mathbb{R}^d$;
- $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ given by $b(x) = (b_i(x))_{i\in\{1,\dots,d\}}$ and $\sigma(x) = (\sigma_{ij}(x))_{i,j\in\{1,\dots,d\}}$ for $x \in \mathbb{R}^d$ are globally Lipschitz;
- $\Psi : \mathbb{R}^d \to \mathbb{R}$, $k : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ are "sufficiently" regular.

Show that F has the following representation

$$F(t,x) = \mathbb{E}\left(\Psi(X_T)e^{-\int_t^T k(s,X_s)ds} + \int_t^T g(s,X_s)e^{-\int_t^s k(r,X_r)dr}ds \,\middle|\, X_t = x\right),$$

for all $t \in [0,T]$ $x \in \mathbb{R}^d$, where

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

and $(W_t)_{t\geq 0}$ is a standard d-dimensional Brownian motion.

Hint: For $s \in [t,T]$, apply Itô to $F(s,X_s)$, and then to

$$G(s,X_s) := e^{-\int_t^s k(r,X_r)dr}F(s,X_s) + \int_t^s e^{-\int_t^r k(u,X_u)du}g(r,X_r)dr.$$

4. (Practical). Once again, suppose that we are in the Black-Scholes setup, and that we would like to find the price C of a European call option with strike $K = 80$ and $T = 1$ year, given that the current price of the underlying is $S_0 = 100$ (we are currently at $t = 0$). Suppose that $r = 0.03$ and $\sigma = 0.2$.

Of course, thanks to the Black-Scholes formula (equation (1) in Question 2 above), we can calculate the exact value of C using the given parameters. What is it?

However, suppose we don't know the exact formula. We therefore instead decide to price the option using a Monte-Carlo method, starting from the general pricing formula from lectures

$$C = \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}(S_T - K)_+ | S_0 = 100\right]$$

where as usual (under the risk-neutral measure $\mathbb{Q}$), $dS_t = rS_t dt + \sigma S_t dW_t$, so that $S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T}$.

(a) Calculate an approximate value of C by a Monte-Carlo method, with N large enough so that we are 95% sure of being within being within 1% of the true value. Justify the calculation of the confidence interval and the choice of N.

(b) We now look to improve the convergence (i.e. reduce the variance) of the previous estimator by the method of antithetic control. Define S_1^- by

$$S_1^- := S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)T - \sigma W_T\right).$$

Show that

$$C = \mathbb{E}\left[e^{-rT}\frac{(S_T - K)_+ + (S_T^- - K)_+}{2}\right].$$

Calculate another approximate value for C with the help of this formula. Compare the precision of this value with the previous method for $N = 1000, 10000$ and 100000 by calculating both the approximate value $\hat{C}$ and the size of the 95% confidence interval for each value of N and each method.

(c) We now look to improve the convergence by importance sampling.

(i) Check that size of the 95% confidence interval as a percentage of the approximate value $\hat{C}$, calculated with either of the above two methods, increases as K increases. Why is this?

(ii) Use the Girsanov Theorem to show that

$$\mathbb{E}[f(W_t)] = \mathbb{E}\left[\exp\left(-\lambda W_t - \frac{\lambda^2}{2}t\right)f(W_t + \lambda t)\right],$$

for all positive measurable functions f, $t \geq 0$.

(iii) Propose a value of λ the allows us to reduce the variance of the estimator for a strike $K = 200$.

Homework

1) $d\beta_t = r\beta_t dt$ risk free

$dX_t = \mu X_t dt + \sigma X_t dB_t$ risky asret

$V_t = a(t)X_t + b(t)\beta_t$ portfolio

$V_0 = V(t=0)$ intial investment

We can put V_0 in Bank $\longrightarrow$ $e^{rT}V_0$ after T
or in The portfolio $\longrightarrow$ $V(T)$

Suppose V is deterministic (its evolution in t) $\longrightarrow$ we know $V(T)$

if $e^{rT}V_0 < V(T)$

$\Rightarrow$ Borrow V_0 from Bank at interest rate r. Invest it on the portfolio Get $V(T)$. Repay The Bank.

risless profit $V(T) - e^{rT}V_0 > 0$

if $e^{rT}V_0 > V(T) \Rightarrow e^{rT}a_0X_0 + \cancel{e^{rT}b_0\beta_0} > a_0X_T + \cancel{b_0e^{rT}}$

go short on ~~[illegible]~~ X_t : get X_0

put X_0 on the Bank : get $e^{rT}X_0$

~~with~~
~~also put your~~

Pay The promise stock. for X_T

You get profit $e^{rT}X_0 - X_T > 0$

Selling short is selling something you dont have but you promise To give

Exercise 2

$$V_t = E_Q\left[e^{-r\theta} h(X_T) \mid F_t\right]$$

Q is risk free measure

contingent claim

$V_T = h(X_T) = (x-K)^+$

1) Discounted price of an stock.

$\tilde{X}_t = e^{-rt} X_t$ " value of stock X_t on future money value "

Itô on $\tilde{X}_t$

$f = e^{-rt}x \quad f_1 = -re^{-rt}x, \; f_2 = e^{-rt}, \; f_{22} = 0$

$df = -re^{-rt}x\,dt + e^{-rt}dx.$

$$d(e^{-rt}X_t) = d\tilde{X}_t = -re^{-rt}X_t dt + e^{-rt}dX_t$$

$$= \cdots\cdots\cdots \left[X_t c\,dt + \sigma X_t dB_t\right]$$

$$= e^{-rt}X_t\left[(c-r)dt + \sigma\, dB_t\right]$$

$$= \tilde{X}_t\, d\underbrace{\left[(c-r)t + \sigma B_t\right]}_{\tilde{B}_t}$$

$$= \tilde{X}_t d\tilde{B}_t = d\tilde{X}_t.$$

Girsanov $\Rightarrow \exists$ and Q s.t $\tilde{B}_t$ is a BM

$$\tilde{X}_t = X_0 e^{-0.5\sigma^2 t + \sigma\tilde{B}_t}$$

is a Martingale in Q. $\quad \tilde{B}_t \subset \sigma(B_t)$

2) Discounted value of $V_t = a_t X_t + b_t \beta_t$

$$\tilde{V}_t = e^{-rt} V_t$$

$$d\tilde{V}_t = -re^{-rt} V_t + e^{-rt} dV_t$$

self financing

$$= -re^{-rt}[a_t X_t + b_t \beta_t] + e^{-rt}[a_t dX_t + b_t d\beta_t]$$

$\downarrow$ $r\beta_t d$

$$= a_t[-re^{-rt} X_t + e^{-rt} dX_t]$$

$$= a_t \, d\tilde{X}_t .$$

$$d\tilde{V}_t = a_t d\tilde{X}_t$$

$$\tilde{V}_t = \tilde{V}_0 + \sigma \int_0^t a_s \tilde{X}_s \, d\tilde{B}_s$$

$\tilde{V}_t$ is Martingale in Q.

3) Martingale property

$$\tilde{V}_t = E_Q[\tilde{V}_T \mid F_t]$$

$$\tilde{V}_T = e^{-rT} V_T = e^{-rT} h(X_T)$$

$$V_t = e^{rt} e^{-rT} E_Q[h(X_T) \mid F_t]$$

$$V_t = e^{-r\theta} E_Q[h(X_T) \mid F_t]$$

$$V_t = E_Q\left[e^{-r\theta} h(X_T) \mid F_t\right]$$

$$= E_Q\left[e^{-r\theta} h\left(X_t e^{(r-0.5\sigma^2)\theta + \sigma(\tilde{B}_T - \tilde{B}_t)}\right) \mid F_t\right]$$

$\tilde{B}_T - \tilde{B}_t$ is ind incr. of F_t, $\sim N(0, \theta)$, X_t is const

$y \; N(0,1)$

$$= e^{-r\theta} \int_{-\infty}^{\infty} h\left(x e^{(r-0.5\sigma^2)\theta + \sigma\theta^{1/2} y}\right) \varphi(y)\, dy$$

$h = \max(x-K, 0)$ $\quad e^{-r\theta}\left(x e^{(r-0.5\sigma^2)\theta + \sigma\theta^{1/2} y} - K\right) > 0$

$$e^{(r-0.5\sigma^2)\theta + \sigma\theta^{1/2} y} > \frac{K}{x}.$$

$$(r-0.5\sigma^2)\theta + \sigma\theta^{1/2} y > \ln\frac{K}{x}.$$

$$y > \frac{\ln\frac{K}{x} - (r-0.5\sigma^2)\theta}{\sigma\theta^{1/2}}$$

$$y < \frac{\ln\frac{x}{K} + (r-0.5\sigma^2)\theta}{\sigma\theta^{1/2}}$$

$$V_t = \underbrace{\int_{-\infty}^{y_2} x e^{-0.5\sigma^2\theta + \sigma\theta^{1/2} y} \varphi(y)\, dy}_{I} - \underbrace{\int_{-\infty}^{y_2} K e^{-r\theta} \varphi(y)\, dy}_{-Ke^{-r\theta}\Phi(y_2)}.$$

$$I) \int_{-\infty}^{y_2} x e^{-0.5\sigma^2\theta + \sigma\theta^{1/2} y} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

$$\frac{1}{2}\left[y^2 - 2\sigma\theta^{1/2} y\right] = \frac{1}{2}\left[y^2 - 2\sigma\theta^{1/2} y + \sigma^2\theta\right] - \frac{\sigma^2\theta}{2}$$

$$= \frac{1}{2}\left[y - \sigma\theta^{1/2}\right]^2 - \frac{\sigma^2\theta}{2}$$

I) $\displaystyle\int_{-\infty}^{y_2} x\, e^{-0.5\sigma^2\theta} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}[y-\sigma\theta^{1/2}]^2 + \frac{\sigma^2\theta}{2}}\, dy$

$$= \int_{-\infty}^{y_2} x\, e^{+\cancel{0.5\sigma^2}\theta - \cancel{0.5\sigma^2}\theta} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}[y-\sigma\theta^{1/2}]^2}\, dy$$

$$u = y - \sigma\theta^{1/2} \qquad du = dy$$

$$= \int_{-\infty}^{u_1} x\, \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}u^2}\, du \qquad * \; u_1 = y_2 - \sigma\theta^{1/2}$$

$$= x\, \Phi(u_1)$$

What a good student!
It is your moment!!! Surprise him

conf int.

$$\hat{C} \pm \frac{1.96\, \sigma}{\sqrt{N}}$$

$$\frac{1.96\, \hat{\sigma}_{(N)}}{\sqrt{N}} \leq 0.01 \cdot C$$

choose $N = N_0$

run MC

for $i: 1 \dots N$

simulate W_i

compute C_i

sum = sum + C_i

squares = squares + C_i^2

end

$$\hat{C} = \frac{1}{N} \text{ sum}$$

$$\hat{\sigma}^2 = \frac{1}{N-1}(\text{square}) - \frac{2\hat{C}}{N-1} \text{ sum} + \frac{N}{N-1}\hat{C}^2$$

check $\frac{1.96\, \hat{\sigma}}{\sqrt{N}} \leq 0.01 \cdot C$

True — done

not true — $N_1 = \frac{1.96^2\, \hat{\sigma}^2}{(0.01)^2\, C^2}$

Generalized Feynman Kac.

$$\begin{cases} \dfrac{\partial F}{\partial t} + \mathcal{L} F - K F + g = 0 & F(t,x) \\ F(T,x) = \Psi(x) \end{cases}$$

Consider $G(s, X_s) := e^{-\int_t^s K(r,X_r)\,dr} F(s, X_s) + \int_t^s e^{-\int_t^r K(u,X_u)\,du} g(r\,X_r)\,dr.$

Itô $dXY = X\,dY + Y\,dX + dX\,dY$. $s \in [t,T]$ $\quad t, T$ fixed.

$$dG = e^{-\int_t^s K\,dr} dF - \left(K\,dr\, e^{-\int_t^s K\,dr}\right) F - \underbrace{\left(K\,dt\, e^{-\int_t^s K\,dr}\right) dF}_{\text{order } dF\,dt \quad \text{drop it}} + e^{-\int_t^r K\,du} g\,dr.$$

Itô To $F(t, X_t)$

$$dF = \frac{\partial F}{\partial s} dt + \frac{\partial F}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dX_t)^2 \qquad dX_t = b(X_t)\,ds + \sigma(X_t)\,dB_t$$

$$= \frac{\partial F}{\partial s} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} \sigma^2(X_t)\,ds + \frac{\partial F}{\partial x}\left[b\,ds + \sigma\,dB_t\right]$$

$$= \underbrace{\left[\frac{\partial F}{\partial s} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2 + \frac{\partial F}{\partial x} b\right]}\,ds + \frac{\partial F}{\partial x}\sigma\,dB_t$$

$$dG = e^{-\int_t^s K\,dr}\left\{\left[\underline{\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2 + \frac{\partial F}{\partial x} b}\right] ds + \frac{\partial F}{\partial x}\sigma\, dB_s\right\}$$

$$-\underline{KF}\, e^{-\int_t^s K\,dr}\, dr + \underline{g}\, e^{-\int_t^s K\,dr}\, dr$$

PDE

$$= \underbrace{e^{-\int_t^s K\,dr}\left[\begin{matrix}\text{PDE}\\ =0\end{matrix}\right]} + e^{-\int_t^s K\,dr}\frac{\partial F}{\partial x}\sigma\, dB_s.$$

$$= e^{-\int_t^s K\,dr}\frac{\partial F}{\partial x}\sigma\, dB_t.$$

$$G(T, X_T) = G(t, X_t) + \int_t^T e^{-\int_t^s K\,dr}\frac{\partial F}{\partial x}\sigma\, dB_t$$

$$\underset{\text{def of } G.}{\mathbb{E}\left[G(T, X_T) \mid X_T = x\right]} = \underbrace{\mathbb{E}\left[G(t, X_t) \mid X_t = x\right]}_{\shortparallel} + \overset{\Sigma\pi\omega}{0}$$

$$\mathbb{E}\left[e^{-\int_t^T K\,dr}\underbrace{F(T, X_T)}_{\Psi(X_T)} + \int_t^T e^{-\int_t^r K\,du} g\, dr \,\Big|\, X_t = x\right] = F(t, X_t)$$

$$F(t, x) = \mathbb{E}\left[\Psi(X_T)\, e^{-\int_t^T K\,ds} + \int_t^T e^{-\int_t^r K\,du} g\, dr \,\Big|\, X_t = x\right]$$

$$\frac{d}{dt}\int_a^t K(s)\,ds$$

$F'(s) = K(s)$ thm Fund calculus

$$\frac{d}{dt}\left[F(t) - F(a) \right] = F'(t) = K(t)$$

$$\frac{d}{dt}\int_t^a K(s)\,ds = \frac{d}{dt}\left[F(a) - F(t) \right]$$

$$= -F'(t) = -K(t)$$

$$\frac{d}{dt}\int_a^{h(t)} K(s)\,ds = K(h(t))\,\frac{dh(t)}{dt}.$$

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