WS 4:

Ex 1

X r.v s.t $E(|X|) < +\infty$. $(\mathcal{F}_t)$ filtration

$\left(Y_t = E(X | \mathcal{F}_t)\right)_{t \geq 0}$ is a $\mathcal{F}_t$ - martingale

$\forall t \geq 0, \quad E(|Y_t|) = E(|E(X | \mathcal{F}_t)|)$
$\leq E(E(|X| \mid \mathcal{F}_t)) = E(|X|) < +\infty$.

$\forall t$, Y_t is $\mathcal{F}_t$ - mesurable by definition
$E(X | \mathcal{F}_t)$ is $\mathcal{F}_t$ - mesurable

$\forall t, \Delta > 0, \ E(Y_{t+\Delta} | \mathcal{F}_t) = Y_t$.

$E(Y_{t+\Delta} | \mathcal{F}_t) = E(E(X | \mathcal{F}_{t+\Delta}) | \mathcal{F}_t)$
as $\mathcal{F}_t \subset \mathcal{F}_{t+\Delta}$ (because $(\mathcal{F}_t)_{t \geq 0}$ is a filtration)
$E(Y_{t+\Delta} | \mathcal{F}_t) = E(X | \mathcal{F}_t) = Y_t$
$\Rightarrow$
$(Y_t)_t$ is $\mathcal{F}_t$ - martingale.

Ex 2: ①

(X_t) $\forall t>s$ $X_t - X_s \perp\!\!\!\perp \sigma(X_u, u \leq s)$
and $E(X_t) = E(X_0)$, $\forall t$

1) (X_t) is a martingale with respect to
$\mathcal{F}_t^X = \sigma(X_s, s \leq t)$

- $\forall t$. $E(|X_t|) < \infty$ (integrable process)
- by def. X_t is $\mathcal{F}_t^X$ measurable.

$\forall s \geq 0$ $E(X_{t+s} | \mathcal{F}_t) = E(X_{t+s} - X_t + X_t | \mathcal{F}_t)$
$= E(X_{t+s} - X_t | \mathcal{F}_t) + E(X_t | \mathcal{F}_t)$
$= E(X_{t+s} - X_t) + X_t$
$= \underset{E(X_0)}{\underset{\shortparallel}{E(X_{t+s})}} - \underset{E(X_0)}{\underset{\shortparallel}{E(X_t)}} + X_t = X_t$.

$(X_t)_{t \geq 0}$ is a martingale with respect to $\mathcal{F}_t^X$.

② $t \geq 0$, $E(X_t^2) < \infty$ then $(X_t^2 - E(X_t^2))_{t \geq 0} = (Y_t)_t$
martingale.

- $E\left(|X_t^2 - E(X^2)| \right) \leq E(|X_t^2|) + E(X_t^2)$
$= 2E(|X_t|^2)$
- $Y_t = f(X_t)$ so $\mathcal{F}_t$ - measurable

X_t is $\mathcal{F}_t$ - measurable by def

Y_t is $\mathcal{F}_t$-measurable as it is a function of X_t.

$$\bullet \; E(Y_{t+s} \mid \mathcal{F}_t) = E(X_{t+s}^2 - E(X_{t+s}^2) \mid \mathcal{F}_t)$$

$$= E(X_{t+s}^2 \mid \mathcal{F}_t) - E(\underbrace{E(X_{t+s}^2)}_{} \mid \mathcal{F}_t)$$

$$= E((X_{t+s} - X_t + X_t)^2 \mid \mathcal{F}_t) - \rightarrow E(X_{t+s}^2)$$

$$= E((X_{t+s} - X_t)^2 + X_t^2 + 2X_t(X_{t+s} - X_t) \mid \mathcal{F}_t)$$
$$- E(X_{t+s}^2)$$

One idea: increments indep

$$= E(X_{t+s} - X_t)^2 + X_t^2 + 2(\underbrace{E((X_{t+s} - X_t)\, x_t \mid \mathcal{F}_t)}_{X_t E(X_{t+s} - X_t)})$$
$$- E(X_{t+s}^2)$$

$$= E(X_{t+s}) - \overset{\shortparallel}{E(X_t)} = 0$$

$$= E(X_{t+s}^2 + X_t^2 - 2X_{t+s}X_t) + X^2$$
$$- E(X_{t+s}^2)$$

$$= E(X_t^2) - 2E(X_t X_{t+s}) + X^2$$

$$E(X_{t+s} X_t) = E(((X_{t+s} - X_t) + X_t)X_t)$$
$$= E((X_{t+s} - X_t)X_t + X_t^2)$$
$$= E((X_{t+s} - X_t)X_t) + E(X_t^2)$$

$$= \overset{0}{\overset{\shortparallel}{E(X_{t+s} - X_t)}}\, E(X_t) + E(X_t^2)$$

$$E(X_{t+s} - X_t) = E(X_{t+s}) - E(X_t) = E(X_0) - E(X_0)$$

So $E(Y_{t+s} \mid \mathcal{F}_t) = -2E(X_t^2) + X_t^2 + E(X_t^2)$

$$= X_t^2 - E(X_t^2) = Y_t.$$

Exy :

$$\mathcal{F}_t^{(B_1, B_2)} := \sigma\{\underbrace{B_1(s)}_{r.v}, \underbrace{B_2(s)}_{r.v}, s \leq t\}$$

$$X(t) = B_1(t) \times B_2(t)$$

- $E|X_t| = E|B_1(t)\, B_2(t)| \underset{\text{indep.}}{=} E|B_1(t)| \cdot E|B_2(t)|$
- $X(t)$ is $\mathcal{F}_t^{(B_1, B_2)}$ measurable as a function of elements $\mathcal{F}_t^{(B_1, B_2)}$ - measurable.
- $\forall s < t \quad E(X_t \mid \mathcal{F}_s) = X_s$.

$$E(X_t \mid \mathcal{F}_s) = E(B_1(t) \cdot B_2(t) \mid \mathcal{F}_s)$$

$$= E\Big([\underbrace{B_1(t) - B_1(s)}_{ind} + B_1(s)]\,[B_2(t) - B_2(s) + B_2(s)] \mid \mathcal{F}_s\Big)$$

$$= E\Big[(B_1(t) - B_1(s))(B_2(t) - B_2(s)) \mid \mathcal{F}_s\Big)$$

$$+ E\Big((\underbrace{B_1(t) - B_1(s)}_{ind})\underbrace{B_2(s)}_{ms} \mid \mathcal{F}_s\Big)$$

$$+ E\left(B_1(s)\left(B_2(t) - B_2(s)\right) \mid \mathcal{F}_s\right)$$
$$+ E\left(B_1(s)\ B_2(s) \mid \mathcal{F}_s\right)$$

$$= E\left(B_1(t) - B_1(s)\right) E\left(B_2(t) - B_2(s)\right)$$
$$+ B_2(s)\ E\left(B_1(t) - B_1(s)\right)$$
$$+ B_1(s)\ E\left(B_2(t) - B_2(s)\right)$$
$$+ B_1(s)\ B_2(s)$$
$$= B_1(s)\ B_2(s)$$

Ex 5 :

B^x BM started in $x \in \mathbb{R}$ $\iff$ $B_0^x = x$.

$T_a = \inf\{t \geq 0,\ B_t = a\}$

hitting time

$B_t^x \sim \mathcal{N}(x, t)$ $B_t^x = x + B_t$ B_t is standard BM

1) B is a martingale (cf ex 2 (1))

A martingale $(M_t)_{t \geq 0}$ is uniformly integrable iff :

$$\lim_{A \to +\infty} \sup_{t \geq 0} E\left(M_t \mid 1_{\{|M_t| \geq A\}}\right) = 0$$

$$\forall t \geq 0, \quad E\left(|B_t^x| \, 1_{\{|B_t^x| \geq A\}}\right) =$$

$$= \int_{\mathbb{R}} |y| \, 1_{\{|y| \geq A\}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy$$

$$= \frac{1}{\sqrt{2\pi t}} \left(\int_A^{+\infty} y \, e^{-\frac{(y-x)^2}{2t}} dy - \int_{-\infty}^{-A} y \, e^{-\frac{(y-x)^2}{2t}} dy \right)$$

$$z = y - x$$

$$= \frac{1}{\sqrt{2\pi t}} \left[\int_{A-x}^{+\infty} (z+x) e^{-\frac{z^2}{2t}} dz - \int_{-\infty}^{-A-x} (z+x) e^{-\frac{z^2}{2t}} dz \right]$$

$$= \frac{1}{\sqrt{2\pi t}} \left(\int_{A-x}^{+\infty} z e^{-\frac{z^2}{2t}} dz + \underbrace{\int_{-\infty}^{-A-x} -z e^{-\frac{z^2}{2t}} dz}_{\geq 0} \right)$$

$$+ x \left[\underbrace{\mathbb{P}(B_t^0 \geq A - x)}_{\geq \text{[illegible]}} - \underbrace{\mathbb{P}(B_t^0 \leq -A - x)}_{\geq -A} \right]$$

$$\frac{1}{\sqrt{2\pi t}} \int_{A-x}^{+\infty} z e^{-\frac{z^2}{2t}} dz = \frac{1}{\sqrt{2\pi t}} \left[-t e^{-\frac{z^2}{2t}} \right]_{A-x}^{+\infty}$$

$$= \sqrt{\frac{t}{2\pi}} \, e^{-\frac{(A-x)^2}{2t}}$$

$$\forall A \geq 0, \quad \sup_{t \geq 0} E\left(|B_t^x| \, 1_{|B_t^x| \geq A}\right) \geq$$

$$\sup_{t\geq 0}\left(\sqrt{\frac{t}{2\pi}}\ \underbrace{e^{-\frac{(A-x)^2}{2t}}}_{\xrightarrow[t\to\infty]{} 1}\right) - 1 = +\infty$$

(B^x) is not uniformly integrable.

(2). We can fix $t_0 > 0$ and consider $(B^x_{t\wedge t_0})_{t\geq 0}$

For any martingale $(M_t)_{t\geq 0}$, the stopped martingale $(M_{t\wedge t_0})_{t\geq 0}$ is uniformly integrable.

Thus, we can use Doob's optional stopping time th ~~time~~ with $(B_{t\wedge t_0})_{t\geq 0}$

For any stopping time T,

$$E(B^x_{T\wedge t_0}) = E(B^x_{0\wedge t_0}) = E(B^x_0) = x$$

$$T = T_a \wedge T_b = \inf\{t\geq 0,\ B_t = a \text{ or } B_t = b\}$$

is a stopping time $\Longrightarrow$

$$E(B^x_{T_a\wedge T_b\wedge t_0}) = x,\ \forall t_0$$

$$E\left(B^{x}_{T_a \wedge T_b} \right) = a\, \mathbb{P}(T_a < T_b) + b\, \mathbb{P}(T_a > T_b)$$

$$= a\left(1 - \mathbb{P}(T_b < T_a)\right) + b\, \mathbb{P}(T_a > T_b)$$

$$= (b-a) \cdot \mathbb{P}(T_b < T_a) + a .$$

× the process is always between a and b

- We want to have :

$$\lim_{t_0 \to +\infty} E\left(B^{x}_{T_a \wedge T_b \wedge t_0} \right) = E\left(B^{x}_{T_a \wedge T_b} \right)$$

we use dominated convergence theorem

As $T_a < +\infty$ a.s (course)

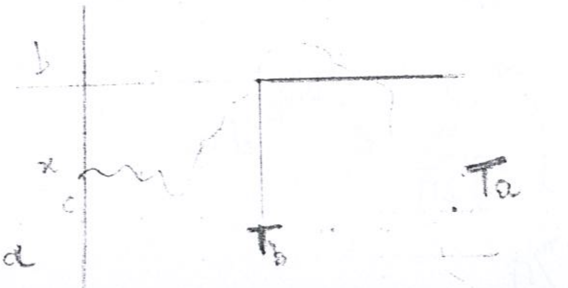
$T_a \wedge T_b < +\infty$ a.s and a.s $\lim\limits_{t_0 \to \infty} B^{x}_{T_a \wedge T_b \wedge t_0} = B^{x}_{T_a \wedge T_b}$

- $\forall t_0, \; \left| B_{T_a \wedge T_b \wedge t_0} \right| \leqslant |a| + |b|$ deterministic

$$E\left[|a| + |b| \right] = |a| + |b| < +\infty .$$

$\Rightarrow$ According to the theorem

$$\lim_{t_0 \to +\infty} \underset{\underset{x}{\shortparallel}}{E\left(B^{x}_{T_a \wedge T_b \wedge t_0} \right)} = E\left(B^{x}_{T_a \wedge T_b} \right)$$



$x = E(B^x_{T_a \wedge T_b}) = a + (b-a)\, \mathbb{P}(T_b < T_a)$

$\Rightarrow \mathbb{P}(T_b^x < T_a^x) = \dfrac{x-a}{b-a}$.

3.a) point (2) of ex 2.

b) $M_t = B_t^2 - t$.

Doobs : $M_{t \wedge T_a \wedge T_b}$ (TCM)

est [illegible] stopping time

$E(M_T \mid F_0) = M_0$.

$E(\underset{\shortparallel}{E(M_T \mid F_0)}) = E(M_0)$

$E(M_T) = E(M_0)$.

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3a) $s \leq t$

$E(B_t^2 - t \mid F_s^B) = E((B_s + (B_t - B_s))^2 - t \mid F_s^B)$

$= B_s^2 + \underbrace{E((B_t - B_s)^2 \mid F_s^B)}_{= t - s} + 2B_s E(B_t - B_s \mid F_s^B)$

$- t = B_s^2 + t - s - t = B_s^2 - s$.

Knowing B_s, $(B_t - B_s)_{t \geq s}$ is a BM.