WORKSHEET 4

In all the exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes the current probability space and $(B_t)_{t\geq 0}$ a (real) Brownian motion.

1. Continuous Martingales

Exercise 1. Let X be a r.v. with finite expectation and $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. Prove that $(\mathbb{E}[X|\mathcal{F}_t])_{t\geq 0}$ is a $(\mathcal{F}_t)_{t\geq 0}$-martingale.

Exercise 2. *Process with independent increments.* Let $(X_t)_{t\geq 0}$ be a integrable process with independent increments, that is for any $t > s$, $X_t - X_s$ is independent of $\sigma(X_u,\ u \leq s)$, such that for any $t \geq 0$, $\mathbb{E}[X_t] = \mathbb{E}[X_0]$. Prove that:

(1) $(X_t)_{t\geq 0}$ is a martingale,
(2) if for any $t \geq 0$, $\mathbb{E}[X_t^2] < \infty$ then $\left(X_t^2 - \mathbb{E}[X_t^2]\right)_{t\geq 0}$ is a martingale,
(3) if, for some $\lambda \in \mathbb{R}$ and for any $t \geq 0$, $\mathbb{E}[e^{\lambda X_t}] < \infty$ then $(Z_t)_{t\geq 0}$ is a martingale, where

$$Z_t = e^{\lambda X_t} / \mathbb{E}\left[e^{\lambda X_t}\right].$$

Exercise 3. Let M be a continuous non negative martingale such that $M_0 = a > 0$ and $\lim_{t\to\infty} M_t = 0$ almost surely.

(1) For $y > 0$, let $T_y = \inf\{t \geq 0,\ M_t = y\}$. Prove that $\mathbb{P}(T_y < \infty) = a/y$.
(2) Prove that $\sup_{t\geq 0} M_t \sim \frac{a}{U}$ where $U \sim \mathcal{U}([0,1])$.

2. Martingale and Brownian motion

Exercise 4. Let B_1 and B_2 be two independent Brownian motions. Prove that the process $X = B_1 B_2$ is a martingale with respect to the filtration $\left(\mathcal{F}_t^{(B_1,B_2)}\right)_{t\geq 0}$.

Exercise 5. Let B be a Brownian motion started in $x \in \mathbb{R}$. Let $a \leq x \leq b$and define $T_a = \inf\{t \geq 0,\ B_t = a\}$ and $T_b = \inf\{t \geq 0,\ B_t = b\}$.

(1) Prove that B is a martingale. Is B uniformly integrable?
(2) Using Doob's optional stopping theorem, prove that

$$\mathbb{P}(T_b < T_a) = \frac{x-a}{b-a}.$$

(3) We suppose now that $x = 0$.
  (a) Prove that the process $\left(B_t^2 - t\right)_{t\geq 0}$ is a $(\mathcal{F}_t^B)_{t\geq 0}$-martingale.
  (b) Use the previous results to show that $\mathbb{E}[T_a \wedge T_b] = |a|b$. What can you say about $\mathbb{E}[T_a]$?

Exercise 6. Let $I = -\inf_{0\leq t\leq T_1} B_t$ where $T_1 = \inf\{t \geq 0,\ B_t = 1\}$. Prove that I is a continuous r.v. with density $f(x) = \frac{1}{(1+x)^2}\mathbf{1}_{x\geq 0}$.

Exercise 7. Let B be a Brownian Motion started in 0 and λ be a real number and define $M_t^\lambda = e^{\lambda B_t - \lambda^2 t/2}$.

(1) Prove that the process $\left(M_t^\lambda\right)_{t\geq 0}$ is a $(\mathcal{F}_t^B)_{t\geq 0}$-martingale.

(2) Let $a \in \mathbb{R}$ and define $T_a = \inf\{t \geq 0, . B_t = a\}$. Prove that $\mathbb{E}[e^{-xT_a}] = e^{-a\sqrt{2x}}$ for any $x \geq 0$.

(3) Prove that for any $a > 0$,

$$\mathbb{P}(\sup_{0\leq s\leq t} M_t^\lambda \geq a) \leq \frac{1}{a}.$$

JUST Apply Doob's Inequality

(4) Use the previous result to show the *exponential inequality* : for any $a > 0$,

$$\mathbb{P}(\sup_{0\leq s\leq t} B_t \geq at) \leq e^{-a^2 t/2}.$$

Worksheet 4

Exercise 1

(X, F_T)

$(E[X|F_t])_{t \geq 0}$ is a F_T Martingale?

* $E\left[\left|E[X|F_T]\right|\right] \leq E\left[\underbrace{E[|X|\,|F_t]}_{}\right]$

$= E[|X|] < \infty$ finite expectation

* $E[X|F_T]$ is adapted (F_T measurable) because X is adapted

* Call $Y_T = E[X|F_T]$

$E[Y_{T+S}|F_T] = E\left[E[X|F_{T+S}]\,|\,F_T\right] \overset{\text{Tower}}{=} E[X|F_T] = Y_T$.

$E[Y_{T+S}|F_T] = Y_T$ Martingale.

Exercise 2. Ind increments.

X_t , $X_t - X_s \perp\!\!\!\perp \sigma(X_u, u \leq s)$ and $E[X_t] = E[X_0]$

X_t integrable.

Any integrable, Ind increment, centered Process is Martingale.

- ~~$E[X_t] < \infty$~~

~~$E[X_t]$~~ $E[|X_t|] < \infty$

- X_t is Adapted (F_t measurable) Natural Filtration

$s > t$

- $$E[X_s | F_t] = E[(X_s - X_t) + X_t | F_t]$$

$$= E[X_s - X_t | F_t] + E[X_t | F_t]$$

independent increments $\qquad + X_t$

$X_s - X_t$ ind $F_t = \sigma(X_t)$.

$$= E[X_s - X_t] + X_t$$

$$= E[X_s] - E[X_t] + X_t$$

$$= \underbrace{E[X_0] - E[X_0]}_{\text{centered.}} + X_t$$

$$= X_t$$

Exercise 2 part b

- $\mathbb{E}|X_t^2 - \mathbb{E}X_t^2| \leq \mathbb{E}|X_t^2| + \mathbb{E}|\mathbb{E}X_t^2| \leq \mathbb{E}|X_t^2| + \mathbb{E}|X_t^2| < \infty$

- $\mathbb{E}[X_t^2 - \mathbb{E}X_t^2 \mid F_s]$

$= \mathbb{E}(X_t \mp X_s)^2 \mid F_s - \mathbb{E}[\mathbb{E}(X_t \mp X_s)^2 \mid F_s]$

$= \mathbb{E}[(X_t - X_s)^2 + X_s^2 + 2(X_t - X_s)X_s \mid F_s] - \mathbb{E}(X_t - X_s + X_s)^2$

if $X_t - X_s \perp\!\!\!\perp F_s \Rightarrow (X_t - X_s)^2 \perp\!\!\!\perp F_s$

X_s is cte wrt F_s

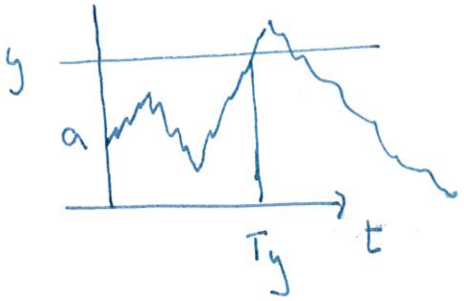
and $\mathbb{E}X_t - X_s = 0$

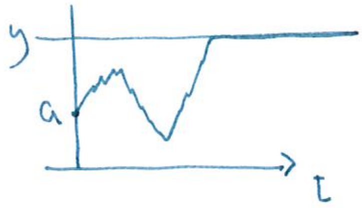
$= \underline{\mathbb{E}(X_t - X_s)^2} + X_s^2 + 0 - \{\underline{\mathbb{E}(X_t - X_s)^2} + \mathbb{E}X_s^2 + \underbrace{2\mathbb{E}(X_t - X_s)X_s}_{=0}\}$

$= X_s^2 - \mathbb{E}X_s^2$ Martingale.

3) M_t

$M_{t \wedge T_y}$ is a UI Martingale.





T_y stopping Time.

$t \wedge T_y$ is stopping Time

$$
\begin{aligned}
\mathbb{E} M_{t \wedge T_y} = \; & \mathbb{E}\, M_t \mathbb{1}_{t < T_y} \mathbb{1}_{T_y < \infty} \\
& + \mathbb{E}\, M_t \mathbb{1}_{t > T_y} \mathbb{1}_{T_y < \infty} \\
& + \mathbb{E}\, M_t \mathbb{1}_{t < T_y} \mathbb{1}_{T_y = \infty}
\end{aligned}
$$

$\lim t \to \infty$

$M_t \to 0$

$$= 0 + \mathbb{E}\, \mathbb{1}_{T_y < \infty}\, y + 0$$

$$= y\, \mathbb{P}(T_y < \infty)$$

by Doob $\mathbb{E} M_{t \wedge T_y} = \mathbb{E} M_0 = a$

$$a = y\, \mathbb{P}(T_y < \infty)$$

$$\mathbb{P}(T_y < \infty) = \frac{y}{a}.$$

Worksheet 4

Exercise 4

Product of $\overset{\text{ind}}{BM}$ is Martingale

$X = B_1 B_2$

$(F_t^{(B_1, B_2)})_{t \geq 0} = \sigma(B_1(s), B_2(s) \quad s \leq t)$

$X(t) = B_1(t) \, B_2(t)$

* $E[|X_t|] = E[|B_1 B_2|] \overset{\text{ind}}{=} E[|B_1|] \, E[|B_2|] < \infty$
* X_t is $F^{(B_1, B_2)}$ Measurable.
* $E[X_t | F_s] = E[B_1(t) B_2(t) | F_s]$

$= E[(B_1(t) - B_1(s) + B_1(s))(B_2(t) - B_2(s) + B_2(s)) | F_s]$

$= E[(B_1(t) - B_1(s))(B_2(t) - B_2(s)) | F_s] + E[(B_1(t) - B_1(s)) B_2(s) | F_s$

$+ E[B_1(s)(B_2(t) - B_2(s)) | F_s] + E[B_1(s) B_2(s) | F_s]$

ind incr

$= E[B_1(t) - B_1(s) | \cancel{F_s}] \, E[B_2(t) - B_2(s) | \cancel{F_s}]$

ind incr

$+ B_2 \underbrace{\cancel{E[B_1(t) - B_1(s)]}}_{0} + B_1(s) \overbrace{\cancel{E[B_2(t) - B_2(s)]}}^{0}$

$+ B_1(s) B_2(s)$

$= B_1(s) \, B_2(s)$

$E[B_1(t) B_2(t) | F_s] = B_1(s) B_2(s)$

Examples

2) Sum of iid centered, integrable

$X_0, X_1, \dots X_n$ centered iid integrables (L_1)

$F_n = \sigma(X_0, X_1, \dots X_n)$

$S_n = \sum_i^n X_i$

Triangular or Minkowski

- $E[|S_n|] = E\left[\left|\sum_i^n X_i\right|\right] \leq E\left[\sum_i^n |X_i|\right] \overset{iid}{=} \sum_i^n E|X_i| < \infty$

$|a+b| \leq |a| + |b|$

$\|a+b\|_{L^p} \leq \|a\|_{L^p} + \|b\|_{L^p}$

- $E[S_{n+1} | F_n] = S_n$

$= E[S_n + X_{n+1} | F_n] = E[S_n | F_n] + E[X_{n+1}|F_n]$

$= S_n + E[X_{n+1}]$

centered

$= S_n$

Example 2

Variance of sum (iid centered integrable)

$M_n = S_n^2 - n\sigma^2$ $\quad E[M_n] = E[S_n^2] - E[n\sigma^2]$

$E[|M_n|] = E[|S_n^2 - n\sigma^2|] \leq E|S_n^2| + E[n\sigma^2| = {}^*$

$= E\left[\sum_{ij}^{n} X_i X_j\right] + n\sigma^2$

$\overset{\text{iid}}{=} E\left[\sum_{i\neq j}^{n} X_i X_j + \sum_{i}^{n} X_i^2\right] + n\sigma^2$

$= E\left[\sum_{i\neq j} X_i X_j\right] + E\left[\sum_{i} X_i^2\right] + n\sigma^2$

Minkoski

$\leq E\sum_{i\neq j}(X_i X_j) + E\sum_{i}(X_i^2) + n\sigma^2$

$\overset{\text{ind}}{=} \cancel{\sum_{i\neq j} E[X_i]E[X_j]} + k\sigma^2 + n\sigma^2$

2nd Method.

$= E\, S_n^2 + n\sigma^2$

$= var\left(\sum_i X_i\right) + n\sigma^2$

$\overset{\text{ind}}{=} \sum_i Var(X_i) + n\sigma^2$

$= n\sigma^2 + n\sigma^2 = 2n\sigma^2$

$< \infty$

$E[M_{n+1} | F_n] = E[S_{n+1}^2 - (n+1)\sigma^2 | F_n]$

$= E[(S_n + X_{n+1})^2 - (n+1)\sigma^2 | F_n]$

$= E[S_n^2 + 2S_n X_{n+1} + X_{n+1}^2 - n\sigma^2 - \sigma^2 | F_n]$

$= -n\sigma^2 - \sigma^2 + 2\overbrace{E[S_n|F_n]}^{S_n}\overbrace{E[X_{n+1}|F_n]}^{E[X_{n+1}]=0} +$

$+ E[S_n^2|F_n] + \underbrace{E[X_{n+1}^2|F_n]}_{E[X_{n+1}^2] = \sigma^2}$

$= -n\sigma^2 + E[S_n^2|F_n]$

$= S_n^2 - n\sigma^2$ ✓

Ex 1.5.1

Any $f(t, B_t)$ is adapted F_t

$B_t^3 - t$

Example 3 Wald's Martingale.

~~X_n~~ Sum iid centered integrable

$$M_n = \frac{\exp(\lambda S_n)}{(E[\exp \lambda X_1])^n}$$

$$\cdot \; E|M_n| = E\left|\frac{\exp \lambda S_n}{(E[\exp \lambda X_1])^n}\right| = E\left|\frac{\exp(\lambda X_1)\exp(\lambda X_2)\dots}{(E[\exp \lambda X_1])^n}\right|$$

$$\overset{ind}{=} \frac{E|e^{\lambda X_1}|\, E|e^{\lambda X_2}| \dots}{(E[e^{\lambda X_1}])^n}$$

$\exp x > 0$ $$\overset{iid}{=} \frac{(E\, e^{\lambda X_1})^n}{(E[e^{\lambda X_1}])^n} = 1.$$

$$\cdot \; E[M_{n+1} | F_n] = E\left[\frac{e^{\lambda S_{n+1}}}{(E[\exp \lambda X_1])^{n+1}} \middle| F_n\right] = E\left[\frac{e^{\lambda S_n} e^{\lambda X_{n+1}}}{(E[e^{\lambda X_1}])^{n+1}} \middle| F_n\right]$$

$$\overset{ind}{=} \frac{E[e^{\lambda X_{n+1}}]}{E[e^{\lambda X_1}]} \frac{e^{\lambda S_n}}{(E[e^{\lambda X_1}])^n}$$

$$\overset{iid}{=} \frac{e^{\lambda S_n}}{E[e^{\lambda X_1}]^n} = M_n.$$

Example 4 Doobs Martingale
Accumulating Data

$$M_n = E[X \mid F_n]$$

$$E[|M_n|] = E[|E[X \mid F_n]|] \leq E[E[|X| \mid F_n]]$$

$$\overset{\text{Rule 2}}{=} E[|X|] < \infty$$

$X \in L_1$.

Rule 2 : $E(X) = E[E(X \mid F)]$

$$E[M_{n+1} \mid F_n] = E[E[X \mid F_{n+1}] \mid F_n] \overset{\text{tower}}{=} E[X \mid F_n]$$

$$= M_n.$$

Proposition

M a Martingale, f convex $\Rightarrow$ $f(M)$ is SubMartingale.

$$M_n = E[M_{n+1} \mid F_n]$$

$$f(M_n) = f[E[M_{n+1} \mid F_n]] \leq E[f(M_{n+1}) \mid F_n]$$

$$f(M_n) \leq E[f(M_{n+1}) \mid F_n]$$

SubMartingale

iii) Doob's Inequality

Y_t a non negative sub Martingale

$$\mathbb{P}\left(\sup |Y_t|^p \geq c\right) \leq \frac{\mathbb{E}(Y_t^p)}{c^p} \qquad c > 0 \quad p \geq 1$$

M_t is non negative Martingale

Any Martingale is also a sub/super Martingale

$$\Rightarrow \mathbb{P}\left(\sup_{0 \leq t \leq T} M_t \geq a\right) \leq \frac{\mathbb{E}[Y_t]}{a} = \frac{1}{a}.$$

$$\mathbb{E} Y_t = e^0 = 1$$

iv)

$$\mathbb{P}(\sup B_t > c) = \mathbb{P}(\sup e^{\lambda B_t} \geq e^{\lambda c})$$

$e^{\lambda B_t}$ is sub Martingale non negative

B_t is Martingale

Doob ineq

$$\leq \frac{\mathbb{E}[e^{\lambda B_T}]}{e^{\lambda c}} = e^{\frac{1}{2}\lambda^2 T - \lambda c}$$

optimize right hand side respect to λ

$$\frac{d}{d\lambda} \frac{1}{2}\lambda^2 t - \lambda c = \lambda t - c = 0$$

$$\Rightarrow \lambda = \frac{c}{T}$$

$$\Rightarrow \mathbb{P}(\sup B_t > c) \leq \underbrace{e^{\frac{1}{2}\frac{c^2 T}{T^2} - \frac{c^2}{T}}}_{e^{-\frac{1}{2}\frac{c^2}{T}}}$$

change $c \to cT$

$$\Rightarrow \mathbb{P}(\sup B_t \geq cT) \leq e^{-\frac{1}{2}c^2 T}$$

7) $M_t = e^{\lambda B_t - \lambda^2 t/2}$, $\mathbb{E}|M_t| = \mathbb{E} M_t = 1 < \infty$

$$\mathbb{E}|M_t| \overset{CS}{\leq} \left(\mathbb{E}(M_t)^2\right)^{1/2} \qquad = e^{-\frac{\lambda^2 t}{2}} \mathbb{E}\, e^{\lambda(B_t - B_s + B_s)}$$

$$= e^{-\frac{\lambda^2 t}{2}} \underbrace{\left(\mathbb{E}\, e^{2\lambda B_t}\right)^{1/2}}_{\text{momentum generator}}$$

$$= e^{-\lambda^2 t/2} \left(e^{\frac{1}{2} 4\lambda^2 (t)}\right)^{1/2}$$

$$= e^{-\lambda^2 t/2 + \lambda^2} < \infty \quad \forall t.$$

$$\mathbb{E}[M_t | F_s] = \mathbb{E}\left[e^{\lambda B_t - \frac{\lambda^2 t}{2}} \,\middle|\, F_s\right]$$

$$= \mathbb{E}\left[e^{\lambda(B_t - B_s + B_s) - \frac{\lambda^2 t}{2}} \,\middle|\, F_s\right]$$

$$= e^{\lambda B_s} e^{-\frac{\lambda^2 t}{2}} \underbrace{\mathbb{E}\left[e^{\lambda(B_t - B_s)}\right]}_{e^{\frac{1}{2}\lambda^2(t-s)}}$$

$$= e^{\lambda B_s - \frac{\lambda^2 s}{2}} = M_s.$$

M_t is UI

because $\sup_t \mathbb{E}|M_t|^p = \sup_t e^{-\frac{\lambda^2 t p}{2}} \underbrace{\mathbb{E} e^{p\lambda B_t}}_{e^{\frac{p^2\lambda^2}{2}t}}$

$= \sup_t e^{\frac{\lambda^2 t}{2}(p^2-p)}$

$< \infty$ just for $p=1$

so $\exists M_\infty$ in $\mathbb{L}^1$

consider $M_{t\wedge T_a}$ is also UI

$\mathbb{E} M_{t\wedge T_a} = \mathbb{E} M_{t\wedge T_a} \mathbb{1}_{T_a<\infty} + \mathbb{E} M_{t\wedge T_a} \mathbb{1}_{T_a=\infty}$

Doob optional $\quad \lim_{t\to\infty} \quad \underbrace{\cancel{e^{-\infty}}_0 e^{\lambda B_\infty}}$

$1 = \mathbb{E} M_{T_a} \mathbb{1}_{T_a<\infty} + 0$

$\frac{1}{M_{T_a}} = \mathbb{P}(T_a<\infty)$

$\frac{1}{e^{\lambda a - \frac{\lambda^2 T_a}{2}}} =$

$1 = \mathbb{E}\, e^{-\frac{\lambda^2 T_a}{2} + \lambda a}$

$e^{-\lambda a} = \mathbb{E}\, e^{-\frac{\lambda^2 T_a}{2}}$

$\frac{\lambda^2}{2} = x \quad \lambda = \sqrt{2x}$

$e^{-\sqrt{2x}\,a} = \mathbb{E}\, e^{-x T_a}$

WORKSHEET 5
WIENER'S INTEGRAL

In all the exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes the current probability space and $(B_t)_{t\geq 0}$ a (real) Brownian motion.

Exercise 1.

(1) Check that the random variable $Y = \int_0^{+\infty} e^{-s}\,\mathrm{d}B_s$ is well-defined.
(2) Give the law of Y.

Exercise 2. Find two admissible functions f and g such that $f \leq g$ and

$$\mathbb{P}\left[\int_0^1 f(s)\,\mathrm{d}B_s > \int_0^1 g(s)\,\mathrm{d}B_s\right] > 0.$$

Exercise 3. Let f be an admissible function. Show that the process $(\int_0^t f(s)\,\mathrm{d}B_s)_{t\geq 0}$ is a Gaussian process. Compute its mean and its covariance.

Exercise 4. Let $(X_t)_{t\geq 0}$ be given by:

$$\forall t \geq 0,\ X_t = \int_0^{t^{1/2}} (2s)^{1/2}\,\mathrm{d}B_s.$$

Show that (X_t) is a Gaussian process. Compute its mean and its covariance. Deduce that X is a Brownian motion.

Exercise 5. Let V_0 be a random variable independent of B and of Gaussian law $\mathcal{N}(0, 1/2)$. We define the process $(V_t)_{t\geq 0}$ (so-called Ornstein-Uhlenbeck stationary process) by:

$$\forall t \geq 0,\ V_t = \exp(-t)V_0 + \int_0^t \exp[-(t-s)]\,\mathrm{d}B_s.$$

(1) Show that $(V_t)_{t\geq 0}$ is a Gaussian process.
(2) For any $a > 0$, prove that $(V_{a+t})_{t\geq 0}$ and $(V_t)_{t\geq 0}$ have the same distribution.

Exercise 6. Let $T > 0$. Show that

$$\lim_{n\to+\infty} \mathbb{E}\left[\left(\sum_{i=1}^{n}(B_{Ti/n} - B_{T(i-1)/n})^2 - T\right)^2\right] = 0.$$

Exercise 7. Let $T > 0$. Show that

$$\int_0^T (1 + \frac{B_t}{n})^n\,\mathrm{d}B_t \xrightarrow[n\to\infty]{\mathrm{L}^2} \int_0^T \exp(B_t)\,\mathrm{d}B_t.$$

Check first that the integrals are well-defined.

Exercise 8. Let $T > 0$. For a given $n \geq 1$, we define the process

$$\forall n \geq 0, \forall t \geq 0, \; B_t^n = \sum_{i=0}^{n-1} B_{Ti/n} \mathbf{1}_{(Ti/n, T(i+1)/n]}(t).$$

(1) Prove that $(B_t^n)_{t \geq 0}$ is a simple process w.r.t. the filtration generated by B.

(2) Show that

$$\lim_{n \to +\infty} \mathbb{E} \int_0^T |B_t^n - B_t|^2 dt = 0.$$

(3) What is the limit, in $L^2(\Omega)$, of

$$\left(\int_0^T B_t^n \, \mathrm{d}B_t \right)_{n \geq 1} ?$$

(4) Prove that

$$B_T^2 = 2 \int_0^T B_t^n \, \mathrm{d}B_t + \sum_{i=1}^{n} \left(B_{Ti/n} - B_{T(i-1)/n} \right)^2$$

(5) By the previous exercise, deduce that

$$B_T^2 = 2 \int_0^T B_t \, \mathrm{d}B_t + T.$$

Worksheet 5

1) $$\int_0^t f(s)\, dB_s \sim N\left(0, \int_0^t f^2(s)\, ds\right)$$

$$Y = \int_0^\infty e^{-s}\, dB_s$$

Well define if $E\left[\int f^2 ds\right] < \infty$

$$= E\left[\int_0^\infty (e^{-s})^2 ds\right] = E\left[\frac{e^{-2s}}{-2}\Big|_0^\infty\right]$$

$$= E\left[\frac{1}{2}\right] = \frac{1}{2} < \infty$$

$$\Rightarrow N\left(0, \frac{1}{2}\right)$$

2) $$\mathbb{P}\left[\int_0^1 f(s)\, dB_s > \int_0^1 g(s)\, dB_s\right] > 0 \qquad \text{s.t } f < g$$

example $f = -k \qquad g = k.$

$$\int_0^1 -k\, dB_s = -k B_1$$

$$\int_0^1 k\, dB_s = k B_1$$

$$\mathbb{P}\left[-kB_1 > kB_1\right] = \mathbb{P}\left[2kB_1 < 0\right] = \frac{1}{2}.$$

Exercise 4

$$X_t = \int_0^{t^{1/2}} (2s)^{1/2} \, dB_s \sim N\left(0, \int_0^{\sqrt{t}} 2s \, ds\right)$$

$$\int_0^{\sqrt{t}} 2s \, ds = s^2 \Big|_0^{\sqrt{t}} = t \quad \Rightarrow N(0, t)$$

if a and b have mean zero

$$\mathrm{cov}(a, a+b) = E[a(a+b)] = E[a^2 + ab] = E[a^2] + E[ab]$$

$$= \mathrm{var}(a) + \mathrm{cov}(a,b)$$

$t < t'$; $\mathrm{cov}\left(\int_0^t, \int_0^{t'}\right) = \mathrm{cov}\left(\int_0^t, \int_0^t + \int_t^{t'}\right)$

$$= \mathrm{cov}\left(\int_0^t, \int_0^t\right) + \mathrm{cov}\left(\int_0^t, \int_{t'}^t\right)$$

but $\int_{t'}^t \perp\!\!\!\perp \int_0^t$

$$= \mathrm{var}\left(\int_0^t\right)$$

$= t$ because $t < t'$

So for $t' < t$ it will be t'

$\Rightarrow \mathrm{cov}(X_t, X_{t'}) = \min(t', t)$.

Exercise 3

Itô $\displaystyle\int_0^t f(s)\,dB_s = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\,(B_{t_i} - B_{t_{i-1}})$

$$S_n = \sum_{i=1}^{n} \underbrace{f(t_{i-1})}_{\text{constant}} \underbrace{(B_{t_i} - B_{t_{i-1}})}_{N(0,\ t_i - t_{i-1})} \sim N$$

as $\Delta_i B$ are independent increments and stationary increments

we have a sum of independent gaussians which is also Gaussian

Now

$$\mu_n = E[S_n] = E\Big[\sum \cdots\Big] = \sum_{i=1}^{n} f(t_{i-1})\,\underbrace{E[\Delta_i B]}_{0} = 0$$

$$\sigma_n^2 = E[S_n^2] = E\Big[\sum_i \sum_j f(t_{i-1}) f(t_{j-1})\,\Delta_i B\,\Delta_j B\Big]$$

$i \neq j \Rightarrow \Delta_i B \perp\!\!\!\perp \Delta_j B$

$$= \sum_i \sum_j f(t_{i-1}) f(t_{j-1})\,E[\Delta_i B]\,E[\Delta_j B] = 0$$

$$i = j \quad = E\sum_i^{n} f^2_{(t_{i-1})}\,(\Delta_i B)^2$$

$$= \sum^{n} f^2_{(t_{i-1})}\,\underbrace{E(\Delta_i B)^2}_{\Delta_i t\, =\, var(\Delta_i B)}$$

$$= \sum^{n} f^2_{t_{i-1}}\,\Delta_i t$$

conv in distribution

$\mu_n \longrightarrow \mu = 0$

$\sigma_n^2 \longrightarrow \sigma = \int_0^t f^2\,ds$

so the characteristic function of S_n goes to

$$N\Big(0, \int_0^t f^2\,ds\Big)$$

because ~~mean~~ L^2 conv. of the sum implies conv in Prob. and dist.

b) $\lim \mathbb{E}\left(\sum \left(B_{T\frac{i}{n}} - B_{\frac{T(i-1)}{n}}\right)^2 - T\right)^2$

$\lim \mathbb{E}\left[\left(\sum \left(B_{\frac{T}{n}}\right)^2\right)^2 + T^2 - 2T \sum (B$

$+ T^2 - 2T \quad n \frac{T}{n}$

$\lim \mathbb{E}\left(\sum \left(\sqrt{\frac{T}{n}} B_1\right)^2\right)^2 + T^2 - 2T^2$

$\lim \mathbb{E}\left(\frac{T}{n} n B_1^2\right)^2 + \quad ''$

$\lim \quad T^2 \underbrace{\mathbb{E} B_1^4}_{\substack{\text{Kurtosis} \\ \text{normal} = 3}} - 2T^2 = T^2$

Exercise 7.

Dominated convergence

Let $\lim f_n = f$ point wise

s.t $|f_n| \leq g(x)$

then $\lim\limits_{n \to \infty} \int |f_n - f| = 0$

also $\lim \int f_n = \int \lim f_n = \int f$

We will show that $\lim \left(1 + \frac{B_t}{n}\right)^n = e^{B_t}$

That in $\left(\int_0^t \left(1 + \frac{B_t}{n}\right)^n dB_t - \int_0^t e^{B_t} dB_t \right)^2$ both are well defined

means $f_n = \left(1 + \frac{B_t}{n}\right)^n - e^{B_t}$

p.w $\lim f_n = 0$

as $(a-b)^2 \geq 0$

$a^2 + b^2 - 2ab \geq 0$ $\quad a^2 + b^2 \geq 2ab$

$2a^2 + 2b^2 \geq (a+b)^2 \geq (a-b)^2$

we can bound f_n by $2\left(1 + \frac{B_t}{n}\right)^{2n} + 2e^{2B_t} = 4e^{2B_t} < \infty$ because well defined

So we can say $\lim \int f_n = \int \lim f_n = \int f$.

$\lim \int \left[\left(1 + \frac{B_t}{n}\right)^n - e^{B_t}\right]^2 dB_t$

$= \int 0 \, dB_t = 0$.

For checking that integrals are well define

$$\int_0^t E[C_s]^2 \, dt < \infty$$

$$E[e^{\lambda z}] = e^{\mu\lambda + \frac{1}{2}\sigma^2\lambda^2}$$

$$E[e^{i\lambda z}] = e^{i\mu\lambda - \frac{1}{2}\sigma^2\lambda^2}$$

$C_s = e^{B_t}$

$$\int_0^t \underbrace{E[e^{2B_t}]}_{\substack{\text{Momentum} \\ \text{Generating} \\ \text{function}}} \, dt = \int_0^t e^{\frac{1}{2}(t)^2(2)^2} \, dt =$$

$$= \int_0^t e^{2t} \, dt < \infty$$

$$C_s = \left(1 + \frac{B_t}{n}\right)^{2n} = e^{2n \log\left(1 + \frac{B_t}{n}\right)}$$

as $1 + x \leq e^x$

$\log(1+x) \leq x$

$$\leq e^{2n \frac{B_t}{n}} = e^{2B_t}$$ (same as before)

so both integrals are well defined.

Exercise 5

$V_0 \sim N(0, \frac{1}{2})$

$$V_t = \underbrace{e^{-t} V_0}_{N(0, \frac{e^{-t}}{2})} + \underbrace{e^{-t} \int_0^t e^s \, d\beta_s}_{N(0, e^{2t} \int_0^t e^{2s} d\beta_s)}$$

sum of normal is normal

if ind $\Rightarrow$ $N(m_1+m_2, var + var_2)$

not ind $\Rightarrow$ $N(m_1+m_2, var + var_2 + 2 cov_{12})$

$\Rightarrow$ V_t is normal

V_{a+t}

$E V_{a+t} = 0 + 0 = 0$

$$var\, V_{a+t} = e^{-2(t+a)} var\, V_0 + e^{-2(t+a)} var \int_0^{(t+a)} e^s \, d\beta_s$$

$$= \frac{e^{-2(t+a)}}{2} + e^{-2(t+a)} \left. \frac{e^{2s}}{2} \right|_0^{t+a}$$

$$= \frac{e^{-2(t+a)}}{2} + e^{-2(t+a)} \frac{1}{2}(e^{2(t+a)} - 1)$$

$= \frac{1}{2}$ ind t, a.

$$cov(V_t, V_{t'}) = \frac{e^{-t-t'}}{2} + e^{-t-t'} \left(\frac{e^{2t} - 1}{2} \right)$$

$\mathrm{cov}(V_t, V_{t'})$

$\mathrm{cov}(A+B, C+D) = \mathrm{cov}\,AC + \mathrm{cov}\,AD + \mathrm{cov}\,BC + \mathrm{cov}\,BD$

$$\mathrm{cov}(e^{-t}V_0, e^{-t'}V_0) = e^{-t-t'}\frac{1}{2}$$

$$\mathrm{cov}\left(e^{-t}V_0, e^{-t'}\int_0^{t'}\right) = e^{-2t'}\cdot 0 \qquad V_0 \perp\!\!\!\perp \int_0^t$$

$$\mathrm{cov}\left(e^{-t}\int_0^t, e^{-t}V_0\right) = 0$$

$$\mathrm{cov}\left(e^{-t}\int_0^t, e^{-t'}\int_0^{t'}\right) = e^{-t-t'}\left(\mathrm{cov}\left(\int_0^t, \int_0^t\right) + \underbrace{\mathrm{cov}\left(\int_0^t, \int_t^{t'}\right)}_{\substack{0 \\ \text{ind incr.}}}\right)$$

$$= e^{-t-t'}\left(\mathrm{var}\int_0^t\right)$$

$$\Rightarrow \mathrm{cov}\,V_t, V_{t'} = \frac{e^{-t-t'}}{2} + e^{-t-t'}\int_0^t e^{2s}\,ds$$

$$= \frac{e^{-t-t'}}{2} + e^{-t-t'}\left(\frac{e^{2t}}{2} - 1\right)$$

$$= e^{-t-t'}\left(\frac{e^{2t}}{2}\right)$$

if $t' = t \Rightarrow \mathrm{cov}\,V_t V_{t'} = \mathrm{var}\,V_t = \frac{1}{2}$

match with

$\mathrm{var}\,V_t =$

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WORKSHEET 6

ITO'S FORMULA

In all the exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes the current probability space and $(B_t)_{t\geq 0}$ a (real) Brownian motion.

Exercise 1. For λ and θ in $\mathbb{R}$, we consider the process

$$\forall t \geq 0,\ X_t = \exp(-\lambda t)\cos(\theta B_t).$$

(1) Compute dX_t for $t \geq 0$.
(2) What are the values of (λ, θ) for which the dt-term in dX_t vanishes?
(3) Deduce $\mathbb{E}[\cos(\theta B_t)]$ for $t \geq 0$.

Exercise 2. For r and σ in $\mathbb{R}$, we consider the process

$$\forall t \geq 0,\ X_t = \exp(rt + \sigma B_t).$$

(1) Compute dX_t for $t \geq 0$.
(2) What are the values of (r, σ) for which the dt-term vanishes?
(3) For the values of r and σ obtained above, show that, for all $0 \leq s < t$,

$$\mathbb{E}\big[X_t|\mathcal{F}_s\big] = X_s,$$

where $\mathcal{F}_s$ is the σ-field generated by $(B_u)_{0\leq u\leq s}$.

Exercise 3. Let n be an integer larger than 1.

(1) Show that

$$\forall t \geq 0,\ B_t^{2n} = 2n\int_0^t B_s^{2n-1}dB_s + n(2n-1)\int_0^t B_s^{2n-2}ds.$$

(2) Deduce that

$$\mathbb{E}(B_1^{2n}) = (2n-1)\mathbb{E}(B_1^{2n-2}).$$

(3) Let Z be an $\mathcal{N}(0,1)$ Gaussian variable. Deduce from the above expression that

$$\mathbb{E}(Z^{2n}) = [(2n)!]/[2^n \times n!].$$

Exercise 4. Show that the following processes are martingales w.r.t. the filtration generated by B:

(1) $\forall t \geq 0,\ X_t = \exp(t/2)\cos(B_t)$.
(2) $\forall t \geq 0,\ Y_t = \exp(t/2)\sin(B_t)$.
(3) $\forall t \geq 0,\ Z_t = (B_t + t)\exp(-B_t - t/2)$.
(4) $\forall t \geq 0,\ W_t = B_t^3 - 3tB_t$.

Exercise 5. Let $(B_t)_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion. Show that $(B_t^4 - 6tB_t^2 + 3t^2)_{t\geq 0}$ is a martingale w.r.t. to the filtration $(\sigma(B_s, s \leq t))_{t\geq 0}$.

Exercise 6. Let $(B_t)_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion and $(b_t)_{t\geq 0}$ be a continuous and $(\mathcal{F}_t)_{t\geq 0}$-adapted process. Set

$$\forall t \geq 0,\ X_t = \int_0^t b_s ds + B_t.$$

We assume that there exist two constants K and λ such that

$$\forall t \geq 0,\ \forall \omega \in \Omega,\ |b_t(\omega)| \leq K,\ b_t(\omega)X_t(\omega) \leq -(\lambda/2)X_t^2(\omega).$$

(1) Show that for all $T \geq 0$, $\sup_{0 \leq t \leq T} \mathbb{E}[X_t^2] < +\infty$.

(2) Applying Itô's formula to $(\exp(\lambda t)X_t^2)_{t\geq 0}$, show

$$\sup_{t\geq 0} \mathbb{E}[X_t^2] < +\infty.$$

Worksheet 6

Exercise 1. $X_t = e^{-\lambda t}\cos(\theta B_t)$

a) $df(t,x) = f_1\,dt + f_2\,dx + \frac{1}{2}[\,f_{11}\,dt^2 + f_{22}\,dx^2 + 2f_{12}\,dt\,dx$

order 2 order 2

$f(t,B_t) = X_t \qquad f(t,x) = e^{-\lambda t}\cos\theta x$

$f_1 = -\lambda X_t \qquad f_2 = -e^{\lambda t}\theta\sin\theta x$

$f_{22} = -\theta^2 e^{\lambda t}\cos\theta x$

plug in $x = B_t$

Keep $dB_t^2 = dt$

$$df(t,B_t) = dX_t = \left[-\lambda X_t - \frac{\theta^2}{2}X_t\right]dt$$

$$- e^{\lambda t}\theta\sin\theta B_t\,dB_t$$

$$dX_t = \left(-\lambda - \frac{\theta^2}{2}\right)X_t\,dt - e^{\lambda t}\theta\sin\theta B_t\,dB_t$$

b) $-\lambda - \frac{\theta^2}{2} = 0$; $\lambda = \frac{-\theta^2}{2}$ //

c) It is a Martingale $\qquad X_t = X_0 + \int_0^t -\theta e^{-\lambda t}\sin\theta B_t\,dB_t$

$E[X_t \mid F_0] = X_0 \qquad = e^{-\lambda t}\cos\theta B_t$

$E[E[X_t \mid F_0]] = E[X_t] = E[X_0]$

$X_0 = e^0\cos(\theta B_0) = 1$

$1 = E[e^{-\lambda t}\cos(\theta B_t)]$

$e^{\lambda t} = E[\cos(\theta B_t)]$

Exercise 2

a) $X_t = e^{rt+\sigma B_t}$ $\quad df(t,x) = f_1\,dt + f_2\,dx + \frac{1}{2}\left[f_{11}\,dt^2 + f_{22}\,dx^2 + 2 f_{12}\,dt\,dx \right]$

$X_t = f(t, B_t)$

$f(t,x) = e^{rt+\sigma x}$ $\quad f_1 = rf \quad f_2 = \sigma f \quad f_{22} = \sigma^2 f$

$$df(t,B_t) = rf\,dt + \sigma f\,dB_t + \frac{1}{2}\sigma^2 f\,dt$$

$$= \left(r + \frac{1}{2}\sigma^2\right) f\,dt + \sigma f\,dB_t$$

$$dX_t = \left(r + \frac{1}{2}\sigma^2\right) X_t\,dt + \sigma X_t\,dB_t.$$

b) $r = -\sigma^2/2$

c) $X_t = X_0 + \int_0^t \sigma X_t\,dB_t$ is Martingale.

$\forall t \quad X_t$ is F_t measurable.

$$E[X_t \mid F_s] = X_0 + E\left[\int_0^t \sigma X_t\,dB_t \,\Big|\, F_s\right]$$

$\int_0^s \subset F_s$ $\quad = X_0 + \int_0^s \sigma X_t\,dB_t + E\left[\int_s^t \sigma X_t\,dB_t \mid F_s\right]$

$\int_s^t \perp\!\!\!\perp F_s$ $\quad = X_0 + \cdots + \underbrace{E\left[\int_s^t \sigma X_t\,dB_t\right]}_{N\left(0,\ \int_s^t E(\sigma^2 X_t^2)\,ds\right)}$

$$= X_0 + \int_0^s \sigma X_t\,dB_t.$$

Exercise 3

a) $f(t,x) = x^{2n}$ $\quad f_1 = 0 \quad f_2 = 2n\, x^{2n-1} \quad f_{22} = 2n(2n-1)\, x^{2n-2}$

$$d f(t,B_t) = \left(f_1 + \frac{1}{2} f_{22}\right) dt + f_2\, dB_t.$$

$$B_t^{2n} - \cancel{B_0^{2n}} = \int_0^t \frac{1}{2}\, 2n(2n-1)\, B_t^{2n-2}\, dt + \int_0^t 2n\, B_t^{2n-1}\, dB_t$$

b) $$E[B_t^{2n}] = n(2n-1)\, E\int_0^t B_t^{2n-2}\, dt + \underbrace{2n\, E\int_0^t B_t^{2n-1}\, dB_t}_{=0}$$

scaling $\quad c^{1/2} B_t \overset{\mathcal{L}}{=} B_{ct}$

$$(t^{1/2} B_1)^{2n-2} \overset{\mathcal{L}}{=} B_t^{2n-2}$$

$$t^{n-1} B_1^{2n-2} \overset{\mathcal{L}}{=} B_t^{2n-2}$$

$$= n(2n-1)\, E\left[B_1^{2n-2}\right] \int_0^t t^{n-1}\, dt$$

$$\underbrace{E[B_t^{2n}]}_{E\left[(t^{1/2} B_1)^{2n}\right]} = \cancel{n}(2n-1)\, E[B_1^{2n-2}]\, \frac{t^n}{\cancel{n}}$$

$$\cancel{t^n}\, E[B_1^{2n}] = (2n-1)\, E[B_1^{2n-2}]\, \cancel{t^n}$$

$$E[B_1^{2n}] = (2n-1)\, E[B_1^{2n-2}]$$

//

c) induction

$$E[Z^0] = \frac{0!}{2^0\,0!} = 1.$$

previous result. $E[Z^{2n}] = (2n-1)\,E[Z^{2n-2}]$

supose it's true for n-1

$$= (2n-1)\,\frac{2(n-1)!}{2^{n-1}\,(n-1)!}$$

$$= \frac{2n\,(2n-1)\,(2n-2)!}{2\;2^{n-1}\;n\,(n-1)!}$$

$$= \frac{2n!}{2^n\,n!}$$

Exercise 4

iii) $X_t = (B_t + t)\, e^{-B_t - t/2} = f(t, B_t)$

$f(t,x) = (x+t)\, e^{-x - t/2}$

$$\int_0^t d f(t, \underset{B_t}{x}) = \int_0^t \left(f_1 + \frac{1}{2} f_{22}\right) dt + \int_0^t f_2 \, dB_t$$

$f(0, B_0) = 0$

$f_1 = e^{-x-t/2} - \frac{1}{2}(x+t)\, e^{-x-t/2}$

$f_2 = e^{-x-t/2} - (x+t)\, e^{-x-t/2}$

$f_{22} = (x+t)\, e^{-x-t/2} - 2 e^{-x-t/2}$

$$X_t = 0 + \int_0^t 0 \, dt + \underbrace{\int_0^t e^{-B_t - t/2} \left(1 - (B_t + t)\right) dB_t}_{\text{itô}}$$

itô is martingale.

Exercise 5

$$X_t = B_t^4 - 6t B_t^2 + 3t^2$$

$$f(t,x) = x^4 - 6tx^2 + 3t^2$$

$$f_1 = -6x^2 + 6t$$

$$f_2 = 4x^3 - 12tx$$

$$f_{22} = 12x^2 - 12t$$

$$X_t - X_0 = \int_0^t -6B_t^2 + 6t + \frac{1}{2}(12B_t^2 - 12t)\, dt$$

$$+ \int_0^t 4B_t^3 - 12tB_t \, dB_t$$

$$= \int_0^t c \, dt + \underbrace{\int_0^t (4B_t^3 - 12tB_t)\,dB_t}_{\text{ITô is Martingale.}}$$

Exercise 6

$$X_t = \int_0^t b_s \, ds + B_t$$

$$X_t^2 = \left(\int_0^t b_s \, ds\right)^2 + B_t^2 + 2B_t \int_0^t b_s \, ds \qquad |b_t| \leq K$$

$$\left| \int_0^t b_s \, ds \right| \leq \int_0^t |b_s| \, ds \leq Kt$$

$$\left| \int_0^t b_s \, ds \right|^2 \leq K^2 t^2$$

so, $E\left|\int_0^t b_s\,ds\right|^2 \leq K^2t^2$

$E[B_t^2] = t.$

$E\left[B_t\int_0^t b_s\,ds\right] = \int_{-\infty}^{\infty} B_t\cdot\int_0^t b_s\,ds\;d\mathbb{P} = \langle B_t, \int b_s\,ds$

$\stackrel{CS}{\leq} (E\,B_t^2)^{1/2}\left(E\left(\int_0^t b_s\,ds\right)^2\right)^{1/2}$

$= t^{1/2}(K^2t^2)^{1/2}$

$E[X_t^2] = K^2t^2 + t + t^{1/2}Kt$

$\sup_t E[X_t^2] = K^2T^2 + T + T^{3/2}K.$

iii) $X_t = \int_0^t b_s\,ds + B_t \qquad f(t,x) = e^{\lambda t}x^2$

$f_1 = \lambda f \quad f_2 = 2e^{\lambda t}x \quad f_{22} = 2e^{\lambda t}$

$e^{\lambda t}X_t^2 - e^{\lambda 0}X_0^2 = f_1\,dt + f_2\,dX_t + \frac{1}{2}[\cdots f_{22}\,dX_t^2 + \cdots]$

$dX_t = b\,dt + dB_t \qquad dX_t^2 = b\,dt^2 + dB_t^2 + 2b\,dt\,dB_t$

$e^{\lambda t}X_t^2 - e^{\lambda 0}X_0^2 = \lambda e^{\lambda t}X_t^2\,dt + 2e^{\lambda t}X_t[b\,dt + dB_t]$

$+\frac{1}{2}[\cdots + 2e^{\lambda t}[b\,dt^2 + dB_t^2 + 2b\,dt\,dB_t]$