Université de Nice - M2 Mathmods - Stochastic Calculus

EXERCISES 7

ITO'S FORMULA AND GIRSANOV FORMULA

Exercise 1. Let $(B_t)_{t\geq 0}$ be a Brownian motion w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ and $(u_t)_{t\geq 0}$ be a continuous and $(\mathcal{F}_t)_{t\geq 0}$-adapted process such that

$$\forall t \geq 0,\ \forall \omega \in \Omega,\ |u_t(\omega)| \leq K,$$

for some constant $K > 0$. We admit the following inequality $\mathbb{E}\left[\exp\left(\int_0^t u_s dB_s\right)\right] \leq \exp(K^2 t/2)$.

(1) Show that the process $\forall t \geq 0,\ M_t = \exp\left(\int_0^t u_s dB_s - \frac{1}{2}\int_0^t u_s^2 ds\right)$, is a martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$. (Use Itô's formula.)

(2) We set $\forall t \geq 0,\ Y_t = -\int_0^t u_s ds + B_t$. Show that the process $(Y_t M_t)_{t\geq 0}$ is a martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$.

Exercise 2. Expand as an Itô process the process

$$\forall t \geq 0,\ X_t = (B_t^1)^2 + (B_t^2)^2 + (B_t^3)^2,$$

where $(B_t^1, B_t^2, B_t^3)_{t\geq 0}$ stands for a Brownian motion of dimension 3.

Exercise 3. Let $(B_t = (B_t^1, B_t^2))_{t\geq 0}$ be two independent Brownian motion w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ and $(u_t)_{t\geq 0}$ and $(v_t)_{t\geq 0}$ be two continuous and $(\mathcal{F}_t)_{t\geq 0}$-adapted processes, bounded by some constant K. Show that

$$\forall t \geq 0, \quad M_t = \exp\left(\int_0^t u_s dB_s^1 + \int_0^t v_s dB_s^2 - \frac{1}{2}\int_0^t (u_s^2 + v_s^2) ds\right),$$

is a martingale.

Exercise 4. Let $(B_t)_{t\geq 0}$ be a Brownian motion w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ and $(u_t)_{t\geq 0}$ and $(v_t)_{t\geq 0}$ ne two continuous and adapted processes such that

$$\forall t \geq 0, \quad \mathbb{E}\int_0^t (u_s^4 + v_s^4) ds < +\infty,$$

show that

$$\left(\left(\int_0^t u_s dB_s\right)\left(\int_0^t v_s dB_s\right) - \int_0^t u_s v_s ds\right)_{t\geq 0}$$

is a martingale.

Exercise 5. Let $(B_t^1, B_t^2)_{t\geq 0}$ be a Brownian motion with values in $\mathbb{R}^2$. We assume that there exists a function u in $\mathcal{C}^{1,2}([0, +\infty) \times \mathbb{R}^2)$, bounded with bounded derivatives, such that

$$(\star) \qquad \forall (t,x) \in [0, +\infty[\times\mathbb{R}^2,\ \frac{\partial u}{\partial t}(t,x) - \frac{1}{2}\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right)(t,x) = 0\ ,\ u(0,x) = h(x)\ .$$

(1) Show that for any $T > 0$ any $x \in \mathbb{R}$, the process $(u(T-t, x+B_t))_{0\leq t\leq T}$ is a martingale.
(2) Deduce that $u(T,x) = \mathbb{E}(h(x + B_T))$.

(3) Deduce that $(\star)$ admits at most one solution $\mathcal{C}^{1,2}$, bounded with bounded derivatives, with $u(0,\cdot)$ as initial condition.

Exercise 6. Let f be a deterministic locally admissible function.

(1) Show that

$$\forall t \geq 0,\ \mathbb{E}\Big[\exp\Big(\int_0^t f_s dB_s\Big)\Big] = \exp\Big(\frac{1}{2}\int_0^t f_s^2 ds\Big).$$

(2) Show that the process

$$\Big(\exp\Big(\int_0^t f_s dB_s - \frac{1}{2}\int_0^t f_s^2 ds\Big)\Big)_{t\geq 0}$$

is a martingale with respect to the natural filtration of B.

Exercise 7. Let $(B_t^1, B_t^2, B_t^3)_{t\geq 0}$ be a three dimensional Brownian motion w.r.t. some filtration $(\mathcal{F}_t)_{t\geq 0}$. For a given vector $(b_1, b_2, b_3) \in \mathbb{R}^3$, we consider the process

$$\forall t \geq 0, \quad X_t = \exp\Big(\sum_{i=1}^3 b_i B_t^i - \frac{1}{2}\sum_{i=1}^3 b_i^2 t\Big).$$

(1) Prove that $(X_t)_{t\geq 0}$ is a square integrable martingale.
(2) Prove that the process $((B_t^1 + B_t^2 - (b_1 + b_2)t)X_t)_{t\geq 0}$ is also a martingale.

Exercise 8. Let $(B_t^1, B_t^2, B_t^3)_{t\geq 0}$ be a three dimensional Brownian motion w.r.t. some filtration $(\mathcal{F}_t)_{t\geq 0}$. For a given matrix σ of size 3×3, we consider the process

$$\forall t \geq 0, \quad X_t = \sigma \times \begin{pmatrix} B_t^1 \\ B_t^2 \\ B_t^3 \end{pmatrix}.$$

Show that the process $(M_t = \sum_{i=1}^3 (X_t^i)^2 - \mathrm{Trace}(\sigma\sigma^*)t)_{t\geq 0}$ is a martingale.

Exercise 9. Let $(B_t)_{t\geq 0}$ be a Brownian and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration. For $\mu \in \mathbb{R}$ et $\sigma > 0$, we set

$$\forall t \geq 0,\ Y_t = \exp\big(\mu t + \sigma B_t\big),$$

referred as Geometric Brownian motion.

(1) We set $r = \mu + \sigma^2/2$ and we define $\forall t \geq 0,\ \widetilde{B}_t = B_t + \sigma^{-1} r t$. What can you say of $(\widetilde{B}_t)_{0\leq t\leq 1}$ under the probability:

$$\forall A \in \mathcal{A},\ \mathbb{Q}(A) = \mathbb{E}\big[\exp\big(-\sigma^{-1} r B_1 - \frac{1}{2}\sigma^{-2} r^2\big)\mathbf{1}_A\big].$$

(2) Show that $(Y_t)_{0\leq t\leq 1}$ is, under the probability $\mathbb{Q}$, a martingale w.r.t. $(\mathcal{F}_t)_{0\leq t\leq 1}$.

Exercise 10. Let $(B_t)_{t\geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration, show that we can define a probabilty $\mathbb{Q}_1$ on $(\Omega, \mathcal{F}_1)$, equivalent to $\mathbb{P}$, such that $(B_t + B_t^3)_{0\leq t\leq 1}$ ($B_t^3 = (B_t)^3$) be a martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ under $\mathbb{Q}_1$. Hint: apply first Itô's formula to $(B_t + B_t^3)_{t\geq 0}$.

Exercise 11. Let $(B_t)_{t\geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration, show that we can define a probabilty $\mathbb{Q}_1$ on $(\Omega, \mathcal{F}_1)$, equivalent to $\mathbb{P}$, such that $((2 + B_t^2)\exp(B_t))_{0\leq t\leq 1}$ ($B_t^2 = (B_t)^2$) be a martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ under $\mathbb{Q}_1$. Hint: apply first Itô's formula to $((2 + B_t^2)\exp(B_t))_{t\geq 0}$.

Worksheet 7

Exercise 1

$$M_t = e^{X_t} = e^{\int_0^t u_t \, dB_t - \frac{1}{2}\int_0^t u_s^2 \, ds}$$

$$f(t, X_t) = e^{X_t}$$

$$df = f_1 \, dt + f_2 \, dX_t + \frac{1}{2} f_{22} \, dX_t^2 + \dots$$

$$dX_t = u_t \, dB_t - \frac{1}{2} u_t^2 \, dt$$

$$dX_t^2 = u_t^2 \, dB_t^2 + \frac{1}{4} u^4 \, dt^2 - u_t^3 \, dt \, dB_t$$

$$df = e^{X_t} dX_t + \frac{1}{2} e^{X_t} dX_t^2$$

$$= e^{X_t}\left[u_t \, dB_t - \frac{1}{2} u_t^2 \, dt + \frac{u_t^2}{2} \, dt \right]$$

$$de^{X_t} = e^{X_t} u_t \, dB_t$$

$$e^{X_t} - \underbrace{e^{X_0}}_{1} = \int_0^t e^{X_t} u_t \, dB_t$$

$$e^{X_t} = 1 + \underbrace{\int_0^t e^{X_t} u_t \, dB_t}_{\text{Ito Martingale}}$$

2nd Method

$$E[|M_t|] = E\left[e^{2\int_0^t u \, dB}\right]^{1/2} \underbrace{E\left[e^{-\frac{2}{2}\int_0^t u^2 ds}\right]^{1/2}}_{\text{fixed}} = e^{-\frac{1}{2}\int u^2 ds}$$

Cauchy Schwartz $E|XY| \le (EX^2)^{1/2}(EY^2)^{1/2}$

$$\le e^{-1/2 \int u^2 ds} \underbrace{\left(E e^{2\int_0^t u \, dB}\right)}_{\text{momentum generating}}^{1/2}$$

$$E e^{\lambda Z} = e^{\lambda m + \frac{1}{2}\sigma^2 \lambda^2} \quad \Big| \quad 2\int u \, dB \text{ is } N\left(0, \int 4u^2 ds\right) \quad \Big| \quad = e^{-\frac{1}{2}\int u^2 ds}\left(e^{2\int u^2 ds}\right)^{1/2} < \infty$$

$$\mathbb{E}[M_t \mid F_s] = e^{\int_0^s u\,dB_s - \frac{1}{2}\int_0^s u^2 ds} \, \underbrace{\mathbb{E}\left[e^{\int_s^t u\,dB_s - \frac{1}{2}\int_s^t u^2 ds} \right]}_{e^{\frac{1}{2}\int_s^t u^2 ds}}$$

because $\int_0^s$ is adapted, $\int^t$ is $\perp\!\!\!\perp$ of F_s

now
$$\mathbb{E}\, e^{\lambda Z} = e^{\lambda m + \frac{1}{2}\lambda^2 (\mathrm{var})}$$
$$\mathrm{var} = e^{\frac{1}{2}\int_s^t u^2 ds} \qquad = e^{\int_0^s u\,dB_s - \frac{1}{2}\int_0^s u^2 ds}$$

2) $d(MY) = Y.dM + M\,dY + dY\,dM$

$$dY = -u\,dt + dB$$
$$dM = Mu\,dB$$

$$= YMu\,dB + (\cancel{-Mu\,dt} + M\,dB)$$
$$+ (\underbrace{-Mu^2 dB\,dt}_{\text{drop}} + \cancel{Mu\,dt})$$

$$= MYu\,dB + M\,dB$$

$$MY = \underbrace{\int M(Yu+1)\,dB}_{\text{Itô}}$$

Exercise 2.

$$X_t = (B^1_t)^2 + (B^2_t)^2 + (B^3_t)^2$$

$f(t, B^i_t)$ $\qquad f(t,x,y,z) = x^2+y^2+z^2$

$$df(t,x,y,z) = f_1\,dt + f_2\,dx + f_3\,dy + f_4\,dz$$
$$+\frac{1}{2}\Big[f_{11}\,dt^2 + f_{22}\,dx^2 + f_{33}\,dy^2 + f_{44}\,dz^4$$
$$+ 2f_{12}\,dt\,dx + 2f_{13}\,dt\,dy + 2f_{14}\,dt\,dz$$
$$+ 2f_{23}\,dx\,dy + 2f_{24}\,dx\,dz + 2f_{34}\,dy\,dz\Big]$$

$\mathbb{E}(dx\,dy)$

$\mathbb{E}[(B^1_{t_i}-B^1_{t_{i-1}})(B^2_{t_i}-B^2_{t_{i-1}})$

$= \mathbb{E}(\ )\,\mathbb{E}(\ ) = 0$

drop $dx\,dy$ $\times$ ...

$f_1=0$	$f_2 = 2x$	$f_3 = 2y$	$f_4 = 2z$
$f_{12}=0$	$f_{22} = 2$	$f_{33} = y$	$f_{44} = z$
$f_{13}=0$	$f_{23}=0$	$f_{34}=0$	
$f_{14}=0$	$f_{24}=0$		

$\mathbb{E}[(B^1_{t_i}-B^1_{t_{i-1}})(B^2_{t_i}-B^2_{t_{i-1}})]^2$

$= dt.\,dt.$ $\quad \Rightarrow dx\,dy = dt^2$

$$dX_t = 0 + 2B^1_t\,dB^1_t + 2B^2_t\,dB^2_t + 2B^3_t\,dB^3_t$$
$$+\frac{1}{2}\Big[0 + 2\,dt + 2\,dt + 2\,dt + 0\Big]$$

$$dX_t = 3\,dt + \sum_{i=1}^{3} 2B^i_t\,dB^i_t$$

$$X_t = 3t + \sum_{i=1}^{3}\int 2B^i_t\,dB^i_t$$

Exercise 3

$$X_t = \int_0^t u \, dB^1_s + \int_0^t v_s \, dB^2_s - \frac{1}{2}\int_0^t (u^2+v^2)\, ds$$

$$f(t,X_t) = e^{X_t} \quad f_1 = 0 \quad f_2 = e^{X_t} \quad f_{22} = e^{X_t}$$

$$df = f_1 dt + f_2 dx + \frac{1}{2}\left[f_{11} dt^2 + f_{22} dx^2 + 2 f_{12} dt\, dx \right]$$

$$dX_t = u\, dB^1_t + v\, dB^2_t - \frac{1}{2} u^2 dt - \frac{1}{2} v^2 dt$$

$$dX_t^2 = \left(u\, dB^1_t + v\, dB^2_t\right)^2 + \left(\frac{1}{2}(u^2+v^2)\, dt\right)^2$$

$$- 2\frac{1}{2}(u^2+v^2)\, dt\, (u\, dB^1_t + v\, dB^2_t)$$

chop unusefull

$$= u^2 (dB^1_t)^2 + v^2 (dB^2_t)^2 + uv\, dB^1_t\, dB^2_t$$

$$de^{X_t} = 0 + e^{X_t}\left[u\, dB^1_t + v\, dB^2_t - \frac{1}{2}(u^2+v^2)\, dt \right]$$

$$+ \frac{1}{2}\left[0 + e^{X_t}\left(u^2 (dB^1_t)^2 + v^2 (dB^2_t)^2 + uv\, dB^1_t\, dB^2_t \right)\right]$$

$$de^{X_t} = e^{X_t}\left[-\frac{1}{2}(u^2+v^2) + \frac{1}{2}\left[u^2+v^2+uv\right]\right] dt$$

$$+ e^{X_t}\left(u\, dB^1_t + v\, dB^2_t\right)$$

$$de^{X_t} = e^{X_t}\left(u\, dB^1_t + v\, dB^2_t\right) + e^{X_t}\, uv\, dt$$

Exercise 3 2^{nd} way

$$s \leq t$$

$$E[M_t | F_s] = E[e^{\int_0^t u dB^1} | F_s]$$

$$\cdot E[e^{\int_0^t v dB} | F_s]$$

$$\cdot E[e^{-\frac{1}{2}\int_0^t u^2+v^2 ds} | F_s]$$

because

$\int_0^s \subset F_s$

$\int_s^t \perp\!\!\!\perp F_s$

$$= e^{\int_0^s u dB_1} e^{\int_0^s v dB_2} e^{-\frac{1}{2}\int_0^t u^2+v^2 ds}$$

$$\cdot \underbrace{E[e^{\int_s^t u dB_1}]}\ \underbrace{E[e^{\int_s^t v dB_2}]}$$

momentum Generators

$$- \quad e^{+\frac{1}{2}\int_s^t u^2 ds} \quad e^{\frac{1}{2}\int_s^t v^2 ds}$$

$$= e^{\int_0^s u dB_1} e^{\int_0^s v dB_2}$$

$$\underbrace{e^{-\frac{1}{2}\int_0^t u^2 ds + \frac{1}{2}\int_s^t u^2 ds}}_{e^{-\frac{1}{2}\int_0^s u^2 ds}} \quad \underbrace{e^{-\frac{1}{2}\int_0^t v^2 ds + \frac{1}{2}\int_s^t v^2 ds}}_{e^{-\frac{1}{2}\int_0^s v^2 ds}}$$

$$= M_s$$

Exercise 4

$$\underbrace{\int_0^t u\,dB}_{X} \underbrace{\int_0^t v\,dB}_{Y} - \int_0^t uv\,ds$$

$$dX = u\,dB$$
$$dY = v\,dB$$
$$dX\,dY = uv\,dt$$

$$d(XY) = Y\,dX + X\,dY + dX\,dY$$

$$d(XY) - dX\,dY = Yu\,dB + Xv\,dB$$

$$\int XY - \int dX\,dY = \underbrace{\int Yu\,dB}_{\text{Itô}} + \underbrace{\int Xv\,dB}_{\text{Itô}}$$

each one must be well defined

$$E[I^2] = E\left[\left(\int_0^t Yu\,dB\right)^2\right] = E\left[\int_0^t Y^2u^2\,d\tau\right]$$

$$= E\left[\int_0^t\int_0^t v^2u^2\,d\tau\,ds\right]$$

Easy way

$$(a^2-b^2)^2 \geq 0$$

$$\cancel{(a+b)^4}$$

$$a^4 - 2a^2b^2 + b^4 \geq 0$$

$$a^4 + b^4 \geq 2a^2b^2$$

Hard way

$$(a+b)^2 = a^2+b^2+2ab$$
$$(a-b)^2 = a^2+b^2-2ab$$

$$4ab = (a+b)^2-(a-b)^2$$

$$4a^2b^2 = (a^2+b^2)^2 - (a^2-b^2)^2 \leq (a^2+b^2)^2 + (a^2-b^2)^2$$
$$= 2a^4+2b^4$$

and

$$(a+b)^2+(a-b)^2 = 2a^2+2b^2$$

$$\Rightarrow a^2b^2 \leq \frac{a^4+b^4}{2}$$

Exercise 4

$$X_t = \underbrace{\left(\int_0^t u \, dB\right)}_{M_1} \underbrace{\left(\int_0^t v \, dB\right)}_{M_2} - \int_0^t u v \, ds$$

in worksheet 4 exercise 4

if $B_1 \perp\!\!\!\perp B_2$

Then $X = B_1 B_2$ is Martingale.

$$d(XY) = X\,dY + Y\,dX + dX\,dY$$

$$X = \int u\,dB \quad dX = u\,dB$$

$$Y = \int v\,dB \quad dY = v\,dB$$

$$d\left(\int u\,dB \int v\,dB\right) - (u\,dB)(v\,dB) = X\,v\,dB + Y\,u\,dB$$

$$\int_0^t u\,dB \cdot \int_0^t v\,dB - \int_0^t uv\,ds = \underbrace{\int X\,v\,dB}_{\text{ITÔ}} + \underbrace{\int Y\,u\,dB}_{\text{ITÔ}}$$

each well defined

$$E\left[\int_0^t X\,v\,dB\right]^2 = E\left[\int_0^t X^2 v^2\,ds\right]$$

$$= E\left[\int_0^t \int_0^t u^2 v^2\,ds\,dr\right]$$

Exercise 6.

$$X_T = e^{\int_0^T f \, dB_s}$$

1) $$I = \int_0^t f_s \, dB_s \sim N\left(0, \int_0^t f_s^2 \, ds\right)$$

$$E\left[e^{\lambda N(\mu, \sigma^2)}\right] = e^{\lambda \mu + \frac{1}{2}\sigma^2 \lambda^2}$$

Momentum Generator

$$\Rightarrow E\left[e^{\int_0^T f \, dB_s}\right] = e^{\frac{1}{2}\int_0^t f^2 \, ds}$$

$$E|X_t| \leq \left(E X_t^2\right)^{1/2} = \left(E\left[e^{2\int_0^t f \, dB_s}\right]\right)^{1/2}$$

$$= \left(E\left[e^{2 N(0, \int_0^t f^2 ds)}\right]\right)^{1/2}$$

$$= e^{\frac{1}{2} 2^2 \int_0^t f^2 ds} < \infty$$

$s \leq t$

$$E[X_t \mid F_s] = E\left[e^{\int_0^t f \, dB - \frac{1}{2}\int_0^t f^2 ds} \mid F_s\right]$$

$$= e^{\int_0^s f \, dB - \frac{1}{2}\int_0^t f^2 ds} \, E\left[\underbrace{e^{\int_s^t f \, dB}}_{\text{ind of } F_s} \mid F_s\right]$$

$$= \text{-- -- -- --} \; E\left[e^{\int_s^t f \, dB}\right]$$

$$= e^{\int_0^s f \, dB - \frac{1}{2}\int_0^t f^2 ds} \; e^{\frac{1}{2}\int_s^t f^2 ds}$$

$$= e^{\int_0^s f \, dB - \frac{1}{2}\int_0^s f^2 ds} = X_s$$

Exercise 5

$$u(T-t, x+B_t)$$

$$du = -u_1\,dt + u_2\,dx + \frac{1}{2}u_{22}\,dx^2 + u_3\,dy + \frac{1}{2}u_{33}\,dy^2 + u_{23}\,dx\,dy$$

$$= u_2\,dx + u_3\,dy + \left[-u_1\,dt + \frac{1}{2}u_{22}\,dx^2 + \frac{1}{2}u_{33}\,dy^2 + u_{23}\,dx\,dy\right]$$

as $B^1 \perp\!\!\!\perp B^2$ $dx\,dy = 0$

$$d(u(T-t, x+B_t)) = \frac{\partial u}{\partial x_1}\,dB^1_t + \frac{\partial u}{\partial x_2}\,dB^2_t + \overbrace{0}^{PDE=0}$$

$\Rightarrow$ Martingale.

now $\int_0^T$

$$\underbrace{u(0, x+B_T)}_{h(0,\,x+B_T)} = \underbrace{u(T, x)}_{\text{known number}} + \int_0^T \frac{\partial u}{\partial x_i}\,dB^i_t$$

$$\mathbb{E}\,h(x+B_T) = u(T,x) + \underset{\text{Itô}}{0}$$

Exercise 9) $dX_t = a\,dt + \theta\,d\beta_t$

$\exists\, u$ s.t $d\hat{\beta}_t = d\beta_t + u\,dt.$

$= a\,dt + \theta\,[d\hat{\beta} - u\,dt]$

$= [a - u\theta]\,dt + \theta\,d\hat{\beta}_t.$

Now given that $u = \frac{r}{\sigma}$

$\Rightarrow dY_t = Y_t \mu\,dt + \sigma Y_t\,d\beta_t.$

$= Y_t \mu\,dt + \sigma Y_t [d\hat{\beta}_t - \frac{r}{\sigma} dt$

$= Y_t [\mu - \frac{r\cancel{\sigma}}{\cancel{\sigma}}]\,dt + \sigma Y_t\,d\hat{\beta}_t$

$\Rightarrow \boxed{\mu = r}$ so Y_t is Mart

we get

$$\frac{dQ}{dP} = M_t = e^{-\int_0^1 \frac{r}{\sigma} d\beta_t - \frac{1}{2}\int_0^1 \frac{r^2}{\sigma^2} dt}$$

$$= e^{-\int_0^1 \frac{\mu}{\sigma} d\beta_t - \frac{1}{2}\int_0^1 \frac{\mu^2}{\sigma^2} dt}.$$

and Y_t is Mart. under Q.

10) $B_t + B_t^3 = f(B_t)$ $\qquad f(x) = x + x^3$

$f_1 = 0$

$f_2 = 1 + 3x^2$

$f_{22} = 6x$

$d(B_t + B_t^3) = (1 + 3B_t^2)\,dB_t$

$+ \frac{1}{2}\, 6B_t\,dt.$ $\Rightarrow B_t + B_t^3 = \underbrace{\int_0^t 3B_t\,dt}_{\text{drift.}} + \underbrace{\int_0^t (1 + 3B_t^2)\,dB_t}_{\text{ITÔ Martingale}}$

$dX_t = a\,dt + \theta\,dB_t$

$d\tilde{B}_t = dB_t + u\,dt$ defines M_t. and $\mathbb{Q}$

$\Rightarrow dX_t = a\,dt + \theta\left[d\tilde{B}_t - u\,dt\right]$

$= [a - u\theta]\,dt + \theta\,d\tilde{B}_t$

choose u st $a - u\theta = 0$ $\Rightarrow dX_t$ is ITÔ (Martingale) in $\mathbb{Q}$

$u = \frac{a}{\theta} = \boxed{\frac{3B_t}{1 + 3B_t^2} = u}$

is ok because denominator is ≥ 0.

$\Rightarrow M_t = e^{-\int u\,dB_s - \frac{1}{2}\int u^2\,ds}$

check well defined $\int u^2\,ds \Rightarrow \int_0^t \mathbb{E}\left[\frac{3B_t}{1 + 3B_t^2}\right]^2 ds$

$= \int_0^t \mathbb{E}\,\frac{9B_t^2}{(1 + 3B_t^2)^2}\,ds$

Girsanov 2

Let $dY_t = \beta_t dt + \theta_t dB_t$

$d\tilde{B}_t = dB_t + u_t dt \quad \Rightarrow \quad M_t = e^{-\int u dB - \frac{1}{2}\int u^2 ds}$

$dY_t = \beta_t dt + \theta_t [d\tilde{B}_t - u_t dt]$

$= \beta_t - u_t \theta_t dt + \theta_t d\tilde{B}_t$

$= \alpha_t dt + \theta_t d\tilde{B}_t$

$\alpha = \beta - u\theta$

Example

$dX_t = a(X_t) dt + dB_t$

$= a(X_t) dt + [d\tilde{B} - u_t dt]$

$= (a-u) dt + d\tilde{B}_t$

$u = -a(X_t)$

define $M_t = \exp\{-\int u dB - \frac{1}{2}\int u^2 ds\}$

$= \exp\{\int a dB - \frac{1}{2}\int a^2 ds\}$

$\tilde{B} + \int a dt$ is BM in Q

Exercise 7

$$X_t = \exp\left(\sum_i^3 b_i B_t^i - \frac{1}{2} \sum_i^3 b_i^2 t \right)$$

$$X_t^2 = \exp\left(2 \sum b_i B_t^i - \sum b_i^2 t \right)$$

$$\begin{aligned}
E[X_t^2] &= \exp\left(-\sum b_i^2 t\right) \mathbb{E}\left[\exp 2 \sum_i^3 b_i B_t^i \right] \\
&= \exp\left(-\sum b_i^2 t\right) \prod_i^3 \underbrace{E \exp\left(2 b_i B_t^i\right)}_{\text{Momentum Generator}} \qquad \operatorname{var} B_t = t \\
&= \exp\left(-\sum b_i^2 t\right) \prod_i^3 \exp\left(\frac{1}{2} t (2 b_i)^2 \right) \\
&= \prod \exp(-b_i^2 t) \prod_i^3 \exp(2 t b_i^2) \\
&= \prod \exp(b_i^2 t).
\end{aligned}$$

Girsanov theorem (Øksendal)

Let $u_t\,dt + dB_t = dY_t$ an Itô process in $\mathbb{P}$

then $M_t = e^{-\int_0^t u\,dB_s - \frac{1}{2}\int_0^t u_s^2\,ds}$ is a Martingale in $\mathbb{P}$

s.t $\frac{d\mathbb{Q}}{d\mathbb{P}} = M_t$.

and in $\mathbb{Q}$ Y_t is an BM

Exercise 10 $B_t^3 - B_t = Y_t$ $\int_0^t dY_t = \int_0^t \frac{1}{2} 6 B_t\,dt + \underbrace{\int_0^t (3B_t^2 - 1)\,dB_t}_{\text{Itô}}$

$Y_t = 3\int_0^t B_t\,dt - B_t + \int_0^t 3B_t^2\,dB_t$

Lipschitz Condition

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq L\,|x-y|$$

- Everywhere differentiable functions with bounded 1rst derivative are Lipschitz Cont.

  Example e^x, x^2 not in $\mathbb{R}$
  yes in compact set $[0,T]$

Growth

$$|b(t,x)| \leq K\,(1+|x|)$$

All linear SDE

$$b(t,x) = c_1(t)\,x + c_2(t)$$

$$|b(t,x)| = |c_1(t)\,x + c_2(t)| \leq \left|\sup_t (c_1, c_2)\,x + \sup_t (c_1, c_2\right.$$

$$= \left|\sup_t (c_1,c_2)\right|\,|1+x|$$

$$\left|\sup_t c_1, c_2\right| < \infty$$

if c_1, c_2 continuous and the set is compact $[0,T]$

Worksheet 9

Growth condition $|b(t,x)| + |\sigma(t,x)| \leq C(1+|x$

Exercise 1

$$dX_t = bX_t\, ds + \sigma X_s\, dB_s$$

a, b must be continuous in the compact set $[0,T]$ and Lipschitz

i) $$dX_t = a(s, X_s)\, ds + b(s, X_s)\, dB_s$$

$$a = bx \qquad b = \sigma x$$

continuous linear functions

~~[illegible]~~

Lipschitz $$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq K|x-y|$$

$$\Rightarrow |bx - by| + |\sigma x - \sigma y| = (|b| + |\sigma|)|x-y|$$

ii) $$f = x_0 e^{(b - \frac{\sigma^2}{2})t + \sigma B_t} \qquad f_1 = \left(b - \frac{\sigma^2}{2}\right) f$$

$$f_2 = \sigma f \qquad f_{22} = \sigma^2 f.$$

$$dX_t = \left(b - \frac{\sigma^2}{2}\right) f + \frac{1}{2}\sigma^2 f \quad dt + \sigma f\, dB_t.$$

$$= bf + \sigma f\, dB_t$$

$$= bX_t + \sigma X_t\, dB_t.$$

Another Method is to consider $d \ln X_t$

Exercise 2

$$dX_t = bX_s\,ds + dB_t$$

$$dX_t = a(s,X_s)\,ds + b(s,X_s)\,dB_t$$

$$a(s,X_s) = bx \qquad b(s,X_s) = 1$$

~~continuous coefficients~~ a, b continuous bounded derivatives => Lipschitz

~~so bounded on compact~~. and cont bounded => growth

Lipschitz:

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)|$$

$$= |bx - by| + |1 - 1| = |b|\,|x-y|$$

so unique strong solution on $[0,T]$

$$|bx| + 1 = |b||x| + 1 \le |b||x| + |b| \le |b|(|x|+1)$$

$$f = X_t = x_0 + \int_0^t bX_s\,ds + B_t$$

$$dX_t = bX_t\,dt + dB_t \qquad dI = -bI\,dt$$

$$I = e^{-bt}$$

$$dX_t - bX_t\,dt = dB_t$$

$$I\,dX_t - bIX_t\,dt = I\,dB_t \qquad \Rightarrow X_t = e^{bt}X_0 + e^{bt}\int_0^t e^{bt}\,dB_t$$

$$d(IX_t) = I\,dB_t$$

$$e^{-bt}X_t - e^{-b0}X_0 = \int e^{-bt}\,dB_t$$

Exercise 3

$$X_t = x + \int_0^t \sin(2X_s)\,ds + \int_0^t (1+\cos^2(X_s))^{1/2}\,dB_s$$

$$dX_t = \sin(2X_s)\,ds + (1+\cos^2(X_s))^{1/2}\,dB_s$$

$$dX_t = a(s, X_s)\,ds + b(s, X_s)\,dB_s$$

bounded in t

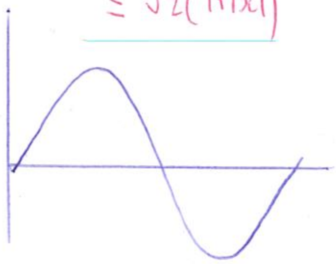
$\sup_t < \infty \Rightarrow$ growth,

$\sin 2x$, $(1+\cos^2(x))^{1/2}$

$|\sin 2x| \le 1 \le 1+|x|$ are cont diff bounded derivative $\Rightarrow$ Lipschitz

$|1+\cos^2 x^{1/2}| \le \sqrt{2} \le \sqrt{2}(1+|x|)$

$$|\sin 2x - \sin 2y| + |(1+\cos^2 x)^{1/2} - (1+\cos^2 y)^{1/2}|$$



$\sin x$ is bounded

$\cos^2 x$ is bounded

$\sqrt{x}$ is bounded in a compact $[0,T]$

$$\Phi = \int_0^x (1+\cos^2(u))^2\,du \qquad \Phi_2' = (1+\cos^2 u)^2 \qquad \Phi_{22} = 2[1+\cos^2 x][-2\cos u \sin u]$$

$$df(X_t) = f_1\,dt + f_2\,dX_t + \frac{1}{2} f_{22}\,dX_t^2 + \dots$$

$$= 0 + [1+\cos(X_t)]^2\,dX_t + \frac{1}{2}\,2[1+\cos^2 u][-2\cos u \sin u]\,dX_t^2$$

$$= [1+\cos X_t]^2\,[\sin 2X_t\,dt + (1+\cos^2 X_t)^{1/2}\,dB_s]$$

$$+ \frac{1}{2}\,2[1+\cos^2 X_t]\underbrace{[-2\cos X_t \sin X_t]}_{-\sin 2X_t}[1+\cos^2 X_t\,dt]$$

$$d\Phi(X_t) = (1+\cos X_t)^{5/2}\,dB_s$$

Exercise 4

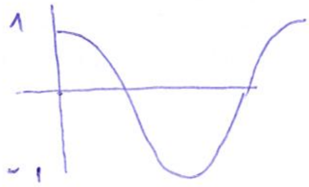
$$dX_t = \frac{-\sin t}{2+\cos t} X_t \, dt + (2+\cos t)\, dB_t .$$

$$a(t,x) = \frac{-\sin t}{2+\cos t} x \qquad b(t,x) = 2+\cos t .$$

are [illegible]

$\sup_t (a,b) < \infty$

$2 + \cos t \neq 0$



$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)|$$

$$= \Big(\underbrace{\left| \frac{-\sin t}{2+\cos t} \right|}_{\leq \frac{1}{2}} + \underbrace{\left| 2+\cos t \right|}_{\leq 3} \Big) |x-y| .$$

$$X_t = (2+\cos t) B_t . \qquad f(t,B_t) = (2+\cos t) x .$$

$$\left(f_1 + \frac{1}{2} f_{22}\right) dt + f_2 \, dB_t . = df .$$

$$\left(-\sin(t) B_t + \frac{1}{2} 0 \, \cancel{B_t}\right) dt + (2+\cos t)\, dB_t .$$

$$dX_t = -\sin t \, B_t \, dt + (2+\cos t)\, dB_t$$

$$\cancel{X} = \cancel{X_t} \qquad \cancel{\frac{X_t}{2+\cos t}} = \cancel{X_t}$$

$$dX_t = \frac{-\sin t}{2+\cos t} \underbrace{(2+\cos t) B_t}_{X_t} + (2+\cos t)\, dB_t$$

Exercise 5.

i) $$X_t = x + \int_0^t s X_s \, ds + \int_0^t e^s \, d\beta_s$$

$$dX_t = t X_t \, dt + e^t \, d\beta_t .$$
$$= a(t, X_t) \, dt + b(t, X_t) \, d\beta_t$$

$$a(t, x) = tx$$
$$b(t, x) = e^t$$ continuous and bounded in compact $[0, T]$

$$| a(t, x) - a(t, y) | + | b(t, x) - b(t, y) |$$

$$= | tx - ty | + | e^t - e^t | \leq T |x - y|$$ Lipschitz.

so $\exists!$ strong solution.

---

ii) $$y_t = 1 + \int_0^t s y_s \, ds \qquad dy_t = t y_t \, dt$$
$$y_t = e^{t^2/2} \qquad y_0 = 1$$

$$d(y^{-1} X) = X \, d(y^{-1}) + y^{-1} dX$$
$$= -X \frac{dy}{y^2} + \frac{1}{y} dX$$
$$= -e^{-t^2} X \, dy + e^{-t^2/2} dX$$
$$= -e^{-t^2} t e^{t^2/2} X \, dt + e^{-t^2/2} [tX \, dt + e^t d\beta_t]$$
$$= \cancel{-t e^{-t^2/2} X \, dt} \cancel{+ t e^{-t^2/2} X \, dt} - e^{+t^2/2} d\beta_t$$
$$= -e^{t - t^2/2} d\beta_t . \Rightarrow \frac{X}{y} \text{ is Martingale.}$$

iii) $$d\left(\frac{X}{y}\right) = -e^{t^2/2}\, dB_t$$

$$X_t = e^{t^2/2} X_0 - e^{t^2/2} \int_0^t e^{t - t^2/2}\, dB_t$$

iv) $$N\left(e^{t^2/2} x_0,\; e^{2t^2/2} \int_0^t e^{2(t - t^2/2)}\, ds\right)$$

Exercise 6

$$M_t = 1 - \int_0^t s^{1/2} M_s \, dB_s$$

$$dM_t = -t^{1/2} M_t \, dB_t$$

$$M_t = f(t, B_t)$$

$$dM_t = f_1 \, dt + f_2 \, dB_t + \frac{1}{2} f_{22} \, dt + \dots$$

$$\begin{cases} f_1 + \frac{1}{2} f_{22} = 0 \\ f_2 = -t^{1/2} f. \end{cases} \Rightarrow \quad f_{22} = t f.$$

$$\Rightarrow \quad f_1 + \frac{t}{2} f = 0.$$

$$X \dot{T} + \frac{t}{2} X T = 0$$

$$\frac{\dot{T}}{T} + \frac{t}{2} = 0 \qquad \frac{\dot{T}}{T} = -\frac{t}{2}$$

$$T = T_0 e^{-t^2/4}$$

!! dB_t

$$\Rightarrow \quad X' T = -t^{1/2} X T \qquad \frac{dX}{X} = -t^{1/2} \, dx$$

$$\frac{X'}{X} = -t^{1/2} \qquad X = X_0 e^{-\int t^{1/2} dB_t}.$$

$$X = X_0 \; e^{-\int_0^t t^{1/2} x}$$

$$F = M_t = XT = M_0 \, e^{-\int_0^t t^{1/2} B_t} e^{-t^2/4} = M_0 \, e^{-\int_0^t t^{1/2} dB_t - \frac{t^2}{4}}$$

ii) $$dM_t = -t^{1/2} M_t \, dB_t.$$

$$= a(t, M_t)\, dt + b(t, M_t)\, dB_t$$

$$a = 0$$

$$b(t, x) = t^{1/2} x.$$

cont. bounded in $[0, T]$

$$|b(t,x) - b(t,y)| = |t^{1/2}(x-y)| \leq T^{1/2}|x-y|$$

$\Rightarrow \exists!$ solution.

$$d\ln(M_t) = 0\, dt + \frac{1}{M_t} dM_t + \frac{1}{2}\left(\frac{-1}{M_t^2}\right) dM_t^2 + \dots$$

$$= \frac{1}{M_t}\left[-t^{1/2} M_t \, dB_t\right] - \frac{1}{2M_t^2}\left[t M_t^2 \, dt\right]$$

$$d\ln(M_t) = -t^{1/2} dB_t - \frac{t}{2} dt.$$

$$\ln M_t - \ln M_0 = -\int_0^t t^{1/2} dB_t - \frac{t^2}{4}$$

$$M_0 = 1$$

$$\ln M_t = 1 - t^2/4 - \int_0^t t^{1/2} dB_t$$

$$M_t = M_0 \, e^{-t^2/4 - \int_0^t t^{1/2} dB_t}.$$

Exercise 6 Continuation

$$dX_t = \cos t \, dt + s^{1/2} X \, dB^1_s$$
$$dM_t = -s^{1/2} M_t \, dB^2_t$$

$$d(e^{s^2/2} X_t M_t) = e^{s^2/2} M dX + e^{s^2/2} X dM + e^{s^2/2} dM dX$$
$$= e^{s^2/2} M \left[\cos t \, dt + s^{1/2} X \, dB^1_s \right]$$
$$+ e^{s^2/2} X \left[-s^{1/2} M \, dB^2_t \right]$$
$$+ e^{s^2/2} \left[\cos t \, dt + s^{1/2} X \, dB^1_s \right] \left[-s^{1/2} M \, dB^2_t \right]$$

$$= e^{s^2/2} M \left[\cos t \, dt \right]$$
$$+ e^{s^2/2} M X s^{1/2} \left[dB^1_t - dB^2_t \right]$$
$$+ e^{s^2/2} \left[\cancel{-s^{1/2} M \cos t \, dt \, dB^2_t} - s M X \, dB^1_t dB^2_t \right]$$

$$e^{s^2/2} X_t M_t = \int e^{s^2/2} M \cos t \, dt \qquad B^1 \perp B^2$$
$$dB^1 dB^2 = 0$$

$$+ \int e^{s^2/2} M X s^{1/2} \, dB^1_t$$
$$- \int e^{s^2/2} M X s^{1/2} \, dB^2_t = \int e^{s^2/2} M \cos t \, dt$$

Exercise 97

$$X_t = 1 - \frac{1}{2}\int_0^t X_s\,ds - \int_0^t Y_s\,dB_s \qquad dX_t = -\frac{X}{2}dt - Y\,dB_t$$

$$Y_t = -\frac{1}{2}\int_0^t Y_s\,ds + \int_0^t X_s\,dB_s \qquad dY_t = -\frac{Y_t}{2}dt + X_t\,dB$$

$$f(t,x) = x^2$$

$$df = f_1\,dt + f_2\,dx + \frac{1}{2}f_{22}\,dx^2 + \dots \qquad (dX_t)^2 = Y^2\,dt + \dots$$

$$= 0 + 2x\,dx + \frac{1}{2}2\,dx^2 \qquad (dY_t)^2 = X_t^2\,dt + \dots$$

$$d(X_t^2) = 2X_t\,dX_t + d(X_t)^2$$

$$d(Y_t^2) = 2Y_t\,dY_t + d(Y_t)^2$$

- $$d(X_t^2) = 2X_t\left[-\frac{X_t}{2}dt - Y\,dB_t\right] + Y^2\,dt$$

$$= (-X^2 + Y^2)\,dt - 2XY\,dB_t$$

- $$d(Y_t^2) = 2Y_t\left[-\frac{Y}{2}dt + X\,dB_t\right] + X_t^2\,dt$$

$$= -Y^2 + X^2\ dt + 2XY\,dB_t$$

$$d(X_t^2) + d(Y_t^2) = 0$$

$$X_t^2 - X_0^2 + Y_t^2 - Y_0^2 = 0$$

$$X_t^2 + Y_t^2 = X_0^2 + Y_0^2 \qquad X_0 = 1,\ Y_0 = 0$$

$$= 1 \checkmark$$

iii) $X_t = \cos\theta_t$

$$dX_t = d\cos\theta_t = -\sin\theta_t \, d\theta_t + \frac{1}{2}\cos\theta_t \, d\theta_t^2$$

$$= -Y_t \, d\theta_t - \frac{1}{2} X_t \, d\theta_t^2$$

$$= -Y_t \, dB_t - \frac{X_t}{2} \, dt$$

$$\Rightarrow \quad d\theta_t = dB_t$$

$$\Rightarrow \quad X_t = \cos B_t$$

$$Y_t = \sin B_t.$$

Exercises Linear SDE

For a Unique solution (strong) $\quad E X_0^2 < \infty$

$a(t,x)$, $b(t,x)$ continuous

$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq k|x-y|$

Lipschitz in 2nd variable

Linear

$a = c_1 x + c_2$

$b = \sigma_1 x + \sigma_2$

continuous

$$|c_1 x + c_2 - (c_1 y - c_2)| + |\sigma_1 x + \sigma_2 - (\sigma_1 y + \sigma_2)|$$

$$= |c_1(x-y)| + |\sigma_1(x-y)| \leq \underbrace{(|c_1| + |\sigma_1|)}_{\text{finite}} |x-y|$$

Every linear has unique strong solution

Geometric Brownian Motion (Linear, Multiplicative Noise, Homogeneous) constant coefficients.

Method 1

$$dX_t = cX_t dt + \sigma X_t dB_t$$

propose $X_t = f(t, B_t)$ unknown

$$df = (f_1 + \tfrac{1}{2} f_{22}) dt + f_2 dB_t.$$

$$\begin{cases} f_1 + \frac{1}{2} f_{22} = cf \\ f_2 = \sigma f \end{cases} \Rightarrow \quad f_{22} = \sigma^2 f. \quad \begin{cases} f_1 = (c - \frac{1}{2}\sigma^2) f \\ f_2 = \sigma f \end{cases}$$

separation of variables $f = T(t)\, h(B_t)$

$$\dot{T} h = \left(c - \frac{1}{2}\sigma^2\right) T h$$

$$T = T_0\, e^{(c-0.5\sigma^2)t}$$

$$T h' = \sigma T h$$

$$h = h_0 e^{\sigma B_t}$$

$$\Rightarrow \quad f = Th = X_0\, e^{(c-0.5\sigma^2)t + \sigma B_t}$$

$$X_t = X_0\, e^{(c-0.5\sigma^2)t + \sigma B_t}$$

Method 2.

$$dX_t = c X_t\, dt + \sigma X_t\, dB_t$$

$$\frac{dX_t}{X_t} = c\, dt + \sigma\, dB_t$$

change variable

$$f(t, X_t) = \ln(X_t)$$

$$df = f_1\, dt + f_2\, dX_t + \frac{1}{2}\left[f_{11}\, dt^2 + f_{22}\, dX_t^2 + 2 f_{12}\, dt\, dX_t \right]$$

Careful when integration of f(g(x

Chain Rule Itô Lemma

watch out!! X_t depends on B_t

$$d \ln X_t \neq \frac{dX_t}{X_t}$$

needs 2nd order corrections

for example

$$d \ln B_t = \frac{dB_t}{B_t} - \frac{(dB_t)^2}{2(B_t)^2}$$

$$f_1 = 0 \quad f_2 = \frac{1}{X_t} \quad f_{22} = -\frac{1}{X_t^2}$$

$$d \ln X_t = \frac{1}{X_t}\left[c X_t\, dt + \sigma X_t\, dB_t \right] + \frac{1}{2}\left(-\frac{1}{X_t^2}\right)\left[\sigma^2 X_t^2\, dt \right]$$

$$d \ln X_t = \left(c - \frac{\sigma^2}{2}\right) dt + \sigma\, dB_t$$

with this change of variables is now Additive Noise

any term depend on the unknown $\ln X_t$.

$$\ln \frac{X_t}{X_0} = \left(c - \frac{\sigma^2}{2}\right) t + \sigma B_t$$

$$X_t = X_0\, e^{(c - \sigma^2/2) t + \sigma B_t}$$

Ornstein Uhlenbeck (Langevin) (Homogeneous, Additive noise, constant coefficients, Linear)

$$dX_t = cX_t\, dt + \sigma\, dB_t$$

$\Delta t = 1$

$$X_{t+1} - X_t = cX_t + \sigma(B_{t+1} - B_t)$$

$$X_{t+1} = (c+1)X_t + \sigma \underbrace{(B_{t+1} - B_t)}_{N(0,1)}$$

Autoregressive order 1.

for 2 Ito Processes X, Y

$$d(XY) = Y\,dX + X\,dY + dX\,dY.$$

and ~~keep~~ drop 2nd order.

Integrating factor

$$Y\,dX_t - cY X_t\, dt = Y\sigma\, dB_t$$

if Y doesnt have random part we recover $d(XY) = Y dX + X dY$ and

$$d(YX_t) = Y\sigma\, dB_t$$

$$Y = e^{-ct} + 0 B_t.$$

$$X_t = e^{ct}\sigma \int_0^t e^{-ct}\, dB_t + X_0 e^{ct}$$

X_t is Gaussian because $\Delta_i B$ is gaussian, $e^{-ct_{i-1}}$ is fixed and S_n is sum of independent gaussians

Suppose $X_0 = 0$

$$S_n = \sum_{i=1}^{n} e^{-ct_{i-1}} \Delta_i B \qquad \mu_n = E[S_n] = 0$$

$$\operatorname{var}(S_n) = \sum_{i=1}^{n} e^{-2ct_{i-1}} \Delta_i t$$

$$\lim_{n\to\infty} \mu_n = 0 \qquad \lim_{n\to\infty} \operatorname{var}(S_n) = \int_0^t e^{-2ct}\, dt$$

$$\Rightarrow X_t \xrightarrow[n\to\infty]{\mathcal{L}} N\left(0, \sigma e^{ct} \int_0^t e^{-2ct}\, dt\right).$$

covariance $\quad \operatorname{cov}(X_s, X_t)$

$$= \operatorname{cov}\left(\sigma e^{ct}\int_0^t e^{-cu}\,du,\ \sigma e^{cs}\int_0^s e^{-cu}\,du\right)$$

$$= \sigma^2 e^{c(t+s)} \operatorname{cov}\left(\int_s^t + \int_0^s e^{-cu}\,du,\ \int_0^s e^{-cu}\,du\right)$$

$$\int_s^t \perp\!\!\!\perp \int_0^s \qquad = \sigma^2 e^{c(t+s)} \operatorname{var}(X_t)$$

$$= \frac{\sigma^2}{2c}\left[e^{c(t+s)} - e^{c(t-s)}\right]$$

~~Exercise~~

<u>2 independent driving Brownian Motions</u>

$$dX_t = cX_t\,dt + \sigma_1 X_t\,dB_t^{(1)} + \sigma_2 X_t\,dB_t^{(2)}$$

Recall $\tilde{B}_t = \frac{1}{(\sigma_1^2+\sigma_2^2)^{1/2}}\left(\sigma_1 B_t^{(1)} + \sigma_2 B_t^{(2)}\right) \qquad B_t^{(1)} \perp\!\!\!\perp B_t^{(2)}$.

$$dX_t = cX_t\,dt + X_t\,d\left[\sigma_1 B_t^{(1)} + \sigma_2 B_t^{(2)}\right]$$

$$= cX_t\,dt + X_t\,d\left[(\sigma_1^2+\sigma_2^2)^{1/2}\tilde{B}_t\right]$$

$$dX_t = cX_t\,dt + (\sigma_1^2+\sigma_2^2)^{1/2} X_t\,d\tilde{B}_t$$

GBM

$$X_t = X_0\, e^{\left[c - 0.5(\sigma_1^2+\sigma_2^2)^{1/2}\right]t + (\sigma_1^2+\sigma_2^2)^{1/2}\tilde{B}_t}$$

Vasicek interest rate

$$dr_t = c[\mu - r_t]dt + \sigma dB_t$$

$$dr_t - c(\mu - r_t)\, dt = \sigma dB_t$$

$$I\, dr_t - Ic(\mu - r_t)\, dt = \sigma dB_t\, I$$

$$I\, dr_t + Ic(r_t - \mu)\, dt = \sigma I\, dB_t$$

$$I\, d(r_t - \mu) + Ic(r_t - \mu)\, dt = \sigma I\, dB_t.$$

I deterministic

$$d(I(r_t - \mu)) = \sigma I\, dB_t$$

$$dI = Ic\, dt \qquad I = e^{ct}.$$

$$r_t - \mu = e^{-ct} I_0^{-1}(r_0 - \mu) + e^{-ct}\sigma \int_0^t e^{ct} dB_t$$

$$r_t = r_0 e^{-ct} + \mu(1 - e^{-ct}) + \sigma e^{-ct} \int_0^t e^{ct} dB_t.$$

for r_0 = constant

$\mu_0 = 0 \Rightarrow$ Langevin

$$E(r_t) = r_0 e^{-ct} + \mu(1 - e^{-ct})$$

$$var(r_t) = \frac{\sigma^2}{2c}(1 - e^{-2ct}) \; ?$$

Example 2.3.5 Itô Exponential $\quad f - f_0 = \int_0^t f \, dB_\tau$

$$dX_t = X_t \, dB_t . \qquad X_t = f(t, B_t)$$

$$df(t, B_t) = f_1 + \frac{1}{2} f_{22} \, dt + f_2 \, dB_t$$

$$\begin{cases} f_1 + \frac{1}{2} f_{22} = 0 \\ f_2 = f \end{cases} \Rightarrow f_{22} = f_2 = f .$$

$$f_1 + \frac{1}{2} f_{22} = 0 \qquad -2 \dot{T} h = h'' T$$

$$\dot{T} h + \frac{1}{2} h'' T = 0 \qquad -2 \frac{\dot{T}}{T} = \frac{h''}{h} = K .$$

$$\cancel{\dot{T} = T_0 e^{-Kt}}$$

$$T = T_0 e^{-\frac{K}{2} t}$$

$$h = h_0 e^{Kx}$$

$$f = f_0 e^{Kx - \frac{K}{2} t} .$$