Chapter 6
Martingale Approach to Pricing and Hedging

In this chapter we present the probabilistic *martingale approach* method to the pricing and hedging of options. In particular, this allows one to compute option prices as the expectations of the discounted option payoffs, and to determine the associated hedging portfolios.

6.1 Martingale Property of the Itô Integral

Recall (Definition 5.3) that an integrable process $(X_t)_{t\in\mathbb{R}_+}$ is said to be a *martingale* with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ if

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s, \qquad 0 \le s \le t.$$

The following result shows that the indefinite Itô integral is a martingale with respect to the Brownian filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. It is the continuous-time analog of the discrete-time Proposition 2.1.

Proposition 6.1. *The indefinite stochastic integral* $\left(\int_0^t u_s dB_s\right)_{t\in\mathbb{R}_+}$ *of a square-integrable adapted process* $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ *is a martingale, i.e.:*

$$\mathbb{E}\left[\int_0^t u_\tau dB_\tau \Big| \mathcal{F}_s\right] = \int_0^s u_\tau dB_\tau, \qquad 0 \le s \le t.$$

Proposition 6.1 is a consequence of Proposition 6.2 below, which shows that

$$\begin{aligned}
\mathbb{E}\left[\int_0^t u_\tau dB_\tau \Big| \mathcal{F}_s\right] &= \mathbb{E}\left[\int_0^\infty \mathbf{1}_{[0,t]}(\tau) u_\tau dB_\tau \Big| \mathcal{F}_s\right] \\
&= \mathbb{E}\left[\int_0^s \mathbf{1}_{[0,t]}(\tau) u_\tau dB_\tau \Big| \mathcal{F}_s\right] \\
&= \int_0^s u_\tau dB_\tau, \qquad 0 \le s \le t.
\end{aligned}$$

↺

Proposition 6.2. *For any* $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ *we have*

$$\mathbb{E}\left[\int_0^\infty u_s dB_s \Big| \mathcal{F}_t\right] = \int_0^t u_s dB_s, \qquad t \in \mathbb{R}_+.$$

In particular, $\int_0^t u_s dB_s$ *is* $\mathcal{F}_t$*-measurable,* $t \in \mathbb{R}_+$.

Proof. The statement is first proved in case u is a simple predictable process, and then extended to the general case, cf. e.g. Proposition 2.5.7 in [89]. For example, for u of the form $u_s := F\mathbf{1}_{[a,b]}(s)$ with F and $\mathcal{F}_a$-measurable random variable and $t \in [a,b]$ we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty u_s dB_s \Big| \mathcal{F}_t\right] &= \mathbb{E}\left[\int_0^\infty F\mathbf{1}_{[a,b]}(s) dB_s \Big| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[F(B_b - B_a) \Big| \mathcal{F}_t\right] \\
&= F\,\mathbb{E}\left[(B_b - B_a) \Big| \mathcal{F}_t\right] \\
&= F(B_t - B_a) \\
&= \int_0^t u_s dB_s, \qquad a \le t \le b.
\end{aligned}$$

On the other hand, when $t \in [0,a]$ we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty u_s dB_s \Big| \mathcal{F}_t\right] &= \mathbb{E}\left[\int_0^\infty F\mathbf{1}_{[a,b]}(s) dB_s \Big| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[F(B_b - B_a) \Big| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[F(B_b - B_a) \Big| \mathcal{F}_a\right] \Big| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[F\,\mathbb{E}\left[B_b - B_a \Big| \mathcal{F}_a\right] \Big| \mathcal{F}_t\right] \\
&= 0 \\
&= \int_0^t u_s dB_s, \qquad 0 \le t \le a.
\end{aligned}$$

The extension from simple processes to square-integrable processes in $L^2_{ad}(\Omega \times \mathbb{R}_+)$ as in Proposition 4.3. Indeed, given $(u^n)_{n\in\mathbb{N}}$ be a sequence of simple predictable processes converging to u in $L^2(\Omega \times [0,T])$, by Fatou's lemma and the continuity of the conditional expectation on L^2 we have:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t u_s dM_s - \mathbb{E}\left[\int_0^\infty u_s dM_s \Big| \mathcal{F}_t\right]\right)^2\right] \\
&\le \liminf_{n\to\infty} \mathbb{E}\left[\left(\int_0^t u^n_s dM_s - \mathbb{E}\left[\int_0^\infty u_s dM_s \Big| \mathcal{F}_t\right]\right)^2\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\left(\mathbb{E}\left[\int_0^\infty u^n_s dM_s - \int_0^\infty u_s dM_s \Big| \mathcal{F}_t\right]\right)^2\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \lim_{n\to\infty} \mathbb{E}\left[\mathbb{E}\left[\left(\int_0^\infty u_s^n dM_s - \int_0^\infty u_s dM_s\right)^2 \Big| \mathcal{F}_t\right]\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^\infty (u_s^n - u_s) dM_s\right)^2\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\int_0^\infty |u_s^n - u_s|^2 ds\right] \\
&= 0,
\end{aligned}$$

where we used the Itô isometry (4.12). □

In particular, since $\mathcal{F}_0 = \{\emptyset, \Omega\}$, this recover the fact that the Itô integral is a centered random variable:

$$\mathbb{E}\left[\int_0^\infty u_s dB_s\right] = \mathbb{E}\left[\int_0^\infty u_s dB_s \Big| \mathcal{F}_0\right] = \int_0^0 u_s dB_s = 0.$$

Examples

1. Given any square-integrable random variable $F \in L^2(\Omega)$, the process $(X_t)_{t\in\mathbb{R}_+}$ defined by $X_t := \mathbb{E}[F \mid \mathcal{F}_t]$, $t \in \mathbb{R}_+$, is a martingale under $\mathbb{P}$, as follows from the "tower property"

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = \mathbb{E}[\mathbb{E}[F \mid \mathcal{F}_t] \mid \mathcal{F}_s] = \mathbb{E}[F \mid \mathcal{F}_s] = X_s, \qquad 0 \leq s \leq t, \quad (6.1)$$

   cf. (16.25) in appendix.
2. Any integrable stochastic process $(X_t)_{t\in\mathbb{R}_+}$ with centered and independent increments is a martingale:

$$\begin{aligned}
\mathbb{E}[X_t|\mathcal{F}_s] &= \mathbb{E}[X_t - X_s + X_s|\mathcal{F}_s] \\
&= \mathbb{E}[X_t - X_s|\mathcal{F}_s] + \mathbb{E}[X_s|\mathcal{F}_s] \\
&= \mathbb{E}[X_t - X_s] + X_s \\
&= X_s, \qquad 0 \leq s \leq t. \qquad (6.2)
\end{aligned}$$

   In particular, the standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ is a martingale because it has centered and independent increments. This fact can also be recovered from Proposition 6.1 since B_t can be written as

$$B_t = \int_0^t dB_s, \qquad t \in \mathbb{R}_+.$$

3. The discounted asset price

$$X_t = X_0 e^{(\mu - r)t + \sigma B_t - \sigma^2 t/2}$$

   is a martingale when $\mu = r$. Indeed we have

$$\begin{aligned}
\mathbb{E}[X_t|\mathcal{F}_s] &= \mathbb{E}[X_0 e^{\sigma B_t - \sigma^2 t/2}|\mathcal{F}_s] \\
&= X_0 e^{-\sigma^2 t/2}\,\mathbb{E}[e^{\sigma B_t}|\mathcal{F}_s] \\
&= X_0 e^{-\sigma^2 t/2}\,\mathbb{E}[e^{\sigma(B_t-B_s)+\sigma B_s}|\mathcal{F}_s] \\
&= X_0 e^{-\sigma^2 t/2+\sigma B_s}\,\mathbb{E}[e^{\sigma(B_t-B_s)}|\mathcal{F}_s] \\
&= X_0 e^{-\sigma^2 t/2+\sigma B_s}\,\mathbb{E}[e^{\sigma(B_t-B_s)}] \\
&= X_0 e^{-\sigma^2 t/2+\sigma B_s} e^{\sigma^2(t-s)/2} \\
&= X_0 e^{\sigma B_s - \sigma^2 s/2} \\
&= X_s, \qquad 0 \le s \le t.
\end{aligned}$$

This fact can also be recovered from Proposition 6.1 since X_t satisfies the equation

$$dX_t = \sigma X_t dB_t,$$

i.e. it can be written as the Brownian stochastic integral

$$X_t = X_0 + \sigma \int_0^t X_u dB_u, \qquad t \in \mathbb{R}_+.$$

4. The discounted value

$$\widetilde{V}_t = e^{-rt} V_t$$

of a self-financing portfolio is given by

$$\widetilde{V}_t = \widetilde{V}_0 + \int_0^t \xi_u dX_u, \qquad t \in \mathbb{R}_+,$$

cf. Lemma 5.1 is a martingale when $\mu = r$ by Proposition 6.1 because

$$\widetilde{V}_t = \widetilde{V}_0 + \sigma \int_0^t \xi_u X_u dB_u, \qquad t \in \mathbb{R}_+,$$

since

$$dX_t = X_t((\mu - r)dt + \sigma dB_t) = \sigma X_t dB_t.$$

Since the Black-Scholes theory is in fact valid for any value of the parameter μ we will look forward to including the case $\mu \neq r$ in the sequel.

6.2 Risk-neutral Measures

Recall that by definition, a risk-neutral measure is a probability measure $\mathbb{P}^*$ under which the discounted asset price $(X_t)_{t\in\mathbb{R}_+} = (e^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale. From the analysis of Section 6.1 it appears that when $\mu = r$, $(X_t)_{t\in\mathbb{R}_+}$ is a martingale and $\mathbb{P}^* = \mathbb{P}$ is risk-neutral.

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In this section we address the construction of a risk-neutral measure in the general case $\mu \neq r$ and for this we will use the Girsanov theorem.

Note that the relation

$$dX_t = X_t((\mu - r)dt + \sigma dB_t)$$

can be rewritten as

$$dX_t = \sigma X_t d\tilde{B}_t,$$

where

$$\tilde{B}_t := \frac{\mu - r}{\sigma} t + B_t, \qquad t \in \mathbb{R}_+.$$

Therefore the search for a risk-neutral measure can be replaced by the search for a probability measure $\mathbb{P}^*$ under which $(\tilde{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

Let us come back to the informal interpretation of Brownian motion via its infinitesimal increments:

$$\Delta B_t = \pm\sqrt{dt},$$

with

$$\mathbb{P}(\Delta B_t = +\sqrt{dt}) = \mathbb{P}(\Delta B_t = -\sqrt{dt}) = \frac{1}{2}.$$

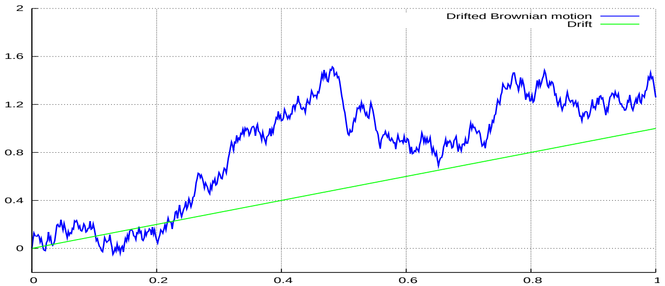

Fig. 6.1: Drifted Brownian path.

Clearly, given $\nu \in \mathbb{R}$, the drifted process $\tilde{B}_t := \nu t + B_t$ is no longer a standard Brownian motion because it is not centered:

$$\mathbb{E}[\nu t + B_t] = \nu t + \mathbb{E}[B_t] = \nu t \neq 0,$$

cf. Figure 6.1. This identity can be formulated in terms of infinitesimal increments as

$$\mathbb{E}[\nu dt + dB_t] = \frac{1}{2}(\nu dt + \sqrt{dt}) + \frac{1}{2}(\nu dt - \sqrt{dt}) = \nu dt \neq 0.$$

In order to make $\nu t + B_t$ a centered process (*i.e.* a standard Brownian motion, since $\nu t + B_t$ conserves all the other properties (i)-(iii) in the definition of Brownian motion, one may change the probabilities of ups and downs, which have been fixed so far equal to $1/2$.

That is, the problem is now to find two numbers $p, q \in [0, 1]$ such that

$$\begin{cases} p(\nu dt + \sqrt{dt}) + q(\nu dt - \sqrt{dt}) = 0 \\ \\ p + q = 1. \end{cases}$$

The solution to this problem is given by

$$p = \frac{1}{2}(1 - \nu\sqrt{dt}) \quad \text{and } q = \frac{1}{2}(1 + \nu\sqrt{dt}).$$

Coming back to Brownian motion considered as a discrete random walk with independent increments $\pm\sqrt{dt}$, we try to construct a new probability measure denoted $\mathbb{P}^*$, under which the drifted process $\tilde{B}_t := \nu t + B_t$ will be a standard Brownian motion. This probability measure will be defined through its density $d\mathbb{P}^*/d\mathbb{P}$ with respect to the historical probability measure $\mathbb{P}$, obtained by taking the product of the above probabilities divided by the reference probability $1/2^N$ corresponding to the symmetric random walk, that is:

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} \simeq \frac{1}{(1/2)^N} \prod_{0<t<T} \left(\frac{1}{2} \mp \frac{1}{2}\nu\sqrt{dt} \right)$$

where 2^N is a normalization factor and $N = T/dt$ is the (infinitely large) number of discrete time steps. Using elementary calculus, this density can be informally shown to converge as follows as N tends to infinity, *i.e.* as the time step $dt = T/N$ tends to zero:

$$\begin{aligned} 2^N \prod_{0<t<T} \left(\frac{1}{2} \mp \frac{1}{2}\nu\sqrt{dt} \right) &= \prod_{0<t<T} \left(1 \mp \nu\sqrt{dt} \right) \\ &= \exp\left(\log \prod_{0<t<T} \left(1 \mp \nu\sqrt{dt} \right) \right) \\ &= \exp\left(\sum_{0<t<T} \log\left(1 \mp \nu\sqrt{dt} \right) \right) \end{aligned}$$

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$$\begin{aligned}
&\simeq \exp\left(\nu \sum_{0<t<T} \mp\sqrt{dt} - \frac{1}{2}\sum_{0<t<T} (\mp\nu\sqrt{dt})^2\right) \\
&= \exp\left(-\nu \sum_{0<t<T} \pm\sqrt{dt} - \frac{1}{2}\nu^2 \sum_{0<t<T} dt\right) \\
&= \exp\left(-\nu B_T - \frac{1}{2}\nu^2 T\right),
\end{aligned}$$

based on the approximations

$$B_T \simeq \sum_{0<t<T} \pm\sqrt{dt} \quad \text{and} \quad T \simeq \sum_{0<t<T} dt.$$

6.3 Girsanov Theorem and Change of Measure

In this section we restate the Girsanov theorem in a more rigorous way, using changes of probability measures. Recall that, given $\mathbb{Q}$ a probability measure on Ω, the notation

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = F$$

means that the probability measure $\mathbb{Q}$ has a density F with respect to $\mathbb{P}$, where F is a non-negative random variable such that $\mathbb{E}[F] = 1$. We also write

$$d\mathbb{Q} = F d\mathbb{P},$$

which is equivalent to stating that

$$\mathbb{E}_{\mathbb{Q}}[\xi] = \int_\Omega \xi(\omega) d\mathbb{Q}(\omega) = \int_\Omega \xi(\omega) F(\omega) d\mathbb{P}(\omega) = \mathbb{E}\left[F\xi\right],$$

where ξ is an integrable random variable. In addition we say that $\mathbb{Q}$ is *equivalent* to $\mathbb{P}$ when $F > 0$ with $\mathbb{P}$-probability one.

Recall that here, $\Omega = \mathcal{C}_0([0, T])$ is the Wiener space and $\omega \in \Omega$ is a continuous function on $[0, T]$ starting at 0 in $t = 0$. Consider the probability $\mathbb{Q}$ defined by

$$d\mathbb{Q}(\omega) = \exp\left(-\nu B_T - \frac{1}{2}\nu^2 T\right) d\mathbb{P}(\omega).$$

Then the process $\nu t + B_t$ is a standard (centered) Brownian motion under $\mathbb{Q}$.

For example, the fact that $\nu T + B_T$ has a standard (centered) Gaussian law under $\mathbb{Q}$ can be recovered as follows:

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[f(\nu T + B_T)] &= \int_\Omega f(\nu T + B_T) d\mathbb{Q} \\
&= \int_\Omega f(\nu T + B_T) \exp\left(-\nu B_T - \frac{1}{2}\nu^2 T\right) d\mathbb{P} \\
&= \int_{-\infty}^{\infty} f(\nu T + x) \exp\left(-\nu x - \frac{1}{2}\nu^2 T\right) e^{-x^2/(2T)} \frac{dx}{\sqrt{2\pi T}} \\
&= \int_{-\infty}^{\infty} f(y) e^{-y^2/(2T)} \frac{dy}{\sqrt{2\pi T}} \\
&= \int_\Omega f(B_T) d\mathbb{P} \\
&= \mathbb{E}_{\mathbb{P}}[f(B_T)].
\end{aligned}
$$

The Girsanov theorem can actually be extended to shifts by adapted processes as follows, cf. e.g. [96], Theorem III-42. Section 14.6 will cover the extension of the Girsanov theorem to jump processes.

Theorem 6.1. *Let $(\psi_t)_{t\in[0,T]}$ be an adapted process satisfying the Novikov integrability condition*

$$
\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |\psi_t|^2 dt\right)\right] < \infty, \tag{6.3}
$$

and let $\mathbb{Q}$ denote the probability measure defined by

$$
\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \psi_s dB_s - \frac{1}{2}\int_0^T \psi_s^2 ds\right).
$$

Then

$$
\tilde{B}_t := B_t + \int_0^t \psi_s ds, \qquad t \in [0, T],
$$

is a standard Brownian motion under $\mathbb{Q}$.

When applied to

$$
\psi_t := \frac{\mu - r}{\sigma},
$$

the Girsanov theorem shows that

$$
\tilde{B}_t := \frac{\mu - r}{\sigma} t + B_t, \qquad t \in [0, T], \tag{6.4}
$$

is a standard Brownian motion under the probability measure $\mathbb{P}^*$ defined by

$$
\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(-\frac{\mu - r}{\sigma} B_T - \frac{(\mu - r)^2}{2\sigma^2} T\right). \tag{6.5}
$$

Hence the discounted price process given by

$$
\frac{dX_t}{X_t} = (\mu - r)dt + \sigma dB_t = \sigma d\tilde{B}_t, \qquad t \in \mathbb{R}_+,
$$

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is a martingale under $\mathbb{P}^*$, hence $\mathbb{P}^*$ is a risk-neutral measure. We obviously have $\mathbb{P} = \mathbb{P}^*$ when $\mu = r$.

6.4 Pricing by the Martingale Method

In this section we give the expression of the Black-Scholes price using expectations of discounted payoffs.

Recall that from the first fundamental theorem of mathematical finance, a continuous market is without arbitrage opportunities if there exists (at least) a risk-neutral probability measure $\mathbb{P}^*$ under which the discounted price process

$$X_t := e^{-rt} S_t, \qquad t \in \mathbb{R}_+,$$

is a martingale under $\mathbb{P}^*$. In addition, when the risk-neutral measure is unique, the market is said to be *complete*.

In case the price process $(S_t)_{s\in[t,\infty)}$ satisfies the equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \qquad t \in \mathbb{R}_+, \qquad S_0 > 0$$

we have

$$S_t = S_0 e^{\sigma B_t - \sigma^2 t/2 + \mu t}, \quad \text{and} \quad X_t = S_0 e^{(\mu - r)t + \sigma B_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+,$$

hence from Section 6.2 the discounted price process is a martingale under the probability measure $\mathbb{P}^*$ defined by (6.5), and $\mathbb{P}^*$ is a martingale measure.

We have

$$dX_t = (\mu - r) X_t dt + \sigma X_t dB_t = \sigma X_t d\tilde{B}_t, \qquad t \in \mathbb{R}_+, \tag{6.6}$$

hence the discounted value $\widetilde{V}_t$ of a self-financing portfolio is written as

$$\begin{aligned}\widetilde{V}_t &= \widetilde{V}_0 + \int_0^t \xi_u dX_u \\ &= \widetilde{V}_0 + \sigma \int_0^t \xi_u X_u d\widetilde{B}_u, \qquad t \in \mathbb{R}_+,\end{aligned}$$

by Lemma 5.1, and becomes a martingale under $\mathbb{P}^*$.

As in Chapter 3, the value V_t at time t of a self-financing portfolio strategy $(\xi_t)_{t\in[0,T]}$ hedging an attainable claim C will be called an *arbitrage price* of the claim C at time t and denoted by $\pi_t(C)$, $t \in [0, T]$.

Proposition 6.3. *Let* $(\xi_t, \eta_t)_{t\in[0,T]}$ *be a portfolio strategy with price*

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \in [0, T],$$

and let C *be a contingent claim, such that*

(i) $(\xi_t, \eta_t)_{t\in[0,T]}$ *is a self-financing portfolio, and*

(ii) $(\xi_t, \eta_t)_{t\in[0,T]}$ *hedges the claim* C*, i.e. we have* $V_T = C$*.*

Then the arbitrage price of the claim C *is given by*

$$V_t = e^{-r(T-t)} \mathbb{E}^*[C|\mathcal{F}_t], \qquad 0 \le t \le T, \tag{6.7}$$

where $\mathbb{E}^*$ *denotes expectation under the risk-neutral measure* $\mathbb{P}^*$*.*

Proof. Since the portfolio strategy $(\xi_t, \eta_t)_{t\in\mathbb{R}_+}$ is self-financing, by Lemma 5.1 and (6.6) we have

$$\widetilde{V}_t = \widetilde{V}_0 + \sigma \int_0^t \xi_u X_u d\tilde{B}_u, \qquad t \in \mathbb{R}_+,$$

which is a martingale under $\mathbb{P}^*$ from Proposition 6.1, hence

$$\begin{aligned}\widetilde{V}_t &= \mathbb{E}^*\left[\widetilde{V}_T \mid \mathcal{F}_t\right] \\ &= e^{-rT}\, \mathbb{E}^*[V_T \mid \mathcal{F}_t] \\ &= e^{-rT}\, \mathbb{E}^*[C \mid \mathcal{F}_t],\end{aligned}$$

which implies

$$V_t = e^{rt}\widetilde{V}_t = e^{-r(T-t)}\, \mathbb{E}^*[C \mid \mathcal{F}_t].$$

□

When the process $(S_t)_{t\in\mathbb{R}_+}$ has the Markov property, the value

$$V_t = e^{-r(T-t)} \mathbb{E}^*[\phi(S_T)|\mathcal{F}_t] = C(t, S_t), \qquad 0 \le t \le T,$$

of the portfolio at time $t \in [0, T]$ can be written from (6.7) as a function $C(t, S_t)$ of t and S_t, and by Proposition 5.3 the function $C(t, x)$ solves the Black-Scholes PDE

$$\begin{cases} rC(t,x) = \dfrac{\partial C}{\partial t}(t,x) + \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 C}{\partial x^2}(t,x) + rx\dfrac{\partial C}{\partial x}(t,x) \\ \\ C(T,x) = \phi(x). \end{cases}$$

In the case of European options with payoff function $\phi(x) = (x - K)^+$ we recover the Black-Scholes formula (5.14), cf. Proposition 5.8, by a probabilistic argument.

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Proposition 6.4. *The price at time t of a European call option with strike K and maturity T is given by*

$$C(t, S_t) = S_t \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-), \qquad t \in [0, T].$$

Proof. The proof of Proposition 6.4 is a consequence of (6.7) and Lemma 6.1 below. Using the relation

$$S_T = S_t e^{r(T-t)+\sigma(\tilde{B}_T - \tilde{B}_t) - \sigma^2(T-t)/2}, \qquad t \in [0, T],$$

by Proposition 6.3 the price of the portfolio hedging C is given by

$$\begin{aligned}
V_t &= e^{-r(T-t)} \, \mathbb{E}^*[C | \mathcal{F}_t] \\
&= e^{-r(T-t)} \, \mathbb{E}^*[(S_T - K)^+ | \mathcal{F}_t] \\
&= e^{-r(T-t)} \, \mathbb{E}^*[(S_t e^{r(T-t)+\sigma(\tilde{B}_T - \tilde{B}_t) - \sigma^2(T-t)/2} - K)^+ | \mathcal{F}_t] \\
&= e^{-r(T-t)} \, \mathbb{E}^*[(x e^{r(T-t)+\sigma(\tilde{B}_T - \tilde{B}_t) - \sigma^2(T-t)/2} - K)^+]_{x=S_t} \\
&= e^{-r(T-t)} \, \mathbb{E}^*[(e^{m(x)+X} - K)^+]_{x=S_t}, \qquad 0 \le t \le T,
\end{aligned}$$

where

$$m(x) = r(T-t) - \sigma^2(T-t)/2 + \log x$$

and $X = \sigma(\tilde{B}_T - \tilde{B}_t)$ is a centered Gaussian random variable with variance

$$\mathrm{Var}\,[X] = \mathrm{Var}\,[\sigma(\tilde{B}_T - \tilde{B}_t)] = \sigma^2 \mathrm{Var}\,[\tilde{B}_T - \tilde{B}_t] = \sigma^2(T-t)$$

under $\mathbb{P}^*$. Hence by Lemma 6.1 below we have

$$\begin{aligned}
V_t &= e^{-r(T-t)} \, \mathbb{E}^*[(e^{m(x)+X} - K)^+]_{x=S_t} \\
&= e^{-r(T-t)} e^{m(S_t)+\sigma^2(T-t)/2} \Phi(v + (m(S_t) - \log K)/v) \\
&\quad - K e^{-r(T-t)} \Phi((m(S_t) - \log K)/v) \\
&= S_t \Phi(v + (m(S_t) - \log K)/v) - K e^{-r(T-t)} \Phi((m(S_t) - \log K)/v) \\
&= S_t \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-),
\end{aligned}$$

$0 \le t \le T$. □

Lemma 6.1. *Let X be a centered Gaussian random variable with variance v^2. We have*

$$\mathbb{E}[(e^{m+X} - K)^+] = e^{m+v^2/2} \Phi(v + (m - \log K)/v) - K \Phi((m - \log K)/v).$$

Proof. We have

$$\begin{aligned}
\mathbb{E}[(e^{m+X} - K)^+] &= \frac{1}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} (e^{m+x} - K)^+ e^{-x^2/(2v^2)} dx \\
&= \frac{1}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} (e^{m+x} - K) e^{-x^2/(2v^2)} dx
\end{aligned}$$

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$$= \frac{e^m}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} e^{x-x^2/(2v^2)} dx - \frac{K}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} e^{-x^2/(2v^2)} dx$$
$$= \frac{e^{m+v^2/2}}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} e^{-(v^2-x)^2/(2v^2)} dx - \frac{K}{\sqrt{2\pi}} \int_{(-m+\log K)/v}^{\infty} e^{-x^2/2} dx$$
$$= \frac{e^{m+v^2/2}}{\sqrt{2\pi v^2}} \int_{-v^2-m+\log K}^{\infty} e^{-x^2/(2v^2)} dx - K\Phi((m - \log K)/v)$$
$$= e^{m+v^2/2}\Phi(v + (m - \log K)/v) - K\Phi((m - \log K)/v).$$

□

Denoting by

$$P(t, S_t) = e^{-r(T-t)}\, \mathbb{E}^*[(K - S_T)^+|\mathcal{F}_t]$$

the price of the put option with strike K and maturity T, we check from Proposition 6.3 that

$$\begin{aligned}
&C(t, S_t) - P(t, S_t) \\
&= e^{-r(T-t)}\, \mathbb{E}^*[(S_T - K)^+|\mathcal{F}_t] - e^{-r(T-t)}\, \mathbb{E}^*[(K - S_T)^+|\mathcal{F}_t] \\
&= e^{-r(T-t)}\, \mathbb{E}^*[(S_T - K)^+ - (K - S_T)^+|\mathcal{F}_t] \\
&= e^{-r(T-t)}\, \mathbb{E}^*[S_T - K|\mathcal{F}_t] \\
&= S_t - e^{-r(T-t)}K.
\end{aligned}$$

This relation is called the *put-call parity*, and it shows that

$$\begin{aligned}
P(t, S_t) &= C(t, S_t) - S_t + e^{-r(T-t)}K \\
&= S_t\Phi(d_+) + e^{-r(T-t)}K - S_t - e^{-r(T-t)}K\Phi(d_-) \\
&= -S_t(1 - \Phi(d_+)) + e^{-r(T-t)}K(1 - \Phi(d_-)) \\
&= -S_t\Phi(-d_+) + e^{-r(T-t)}K\Phi(-d_-).
\end{aligned}$$

6.5 Hedging Strategies

In the next proposition we compute a self-financing hedging strategy leading to an arbitrary square-integrable random variable C admitting a stochastic integral representation formula of the form

$$C = \mathbb{E}^*[C] + \int_0^T \zeta_t d\tilde{B}_t, \tag{6.8}$$

where $(\zeta_t)_{t\in[0,t]}$ is a square-integrable adapted process. Consequently, the mathematical problem of finding the predictable representation (6.8) of a given random variable has important applications in finance. For example we have

$$B_T^2 = T + 2\int_0^T B_t dB_t,$$

and

$$B_T^3 = 3\int_0^T (T - t + B_t^2) dB_t,$$

cf. Exercise 4.2.

Recall that the risky asset follows the equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \qquad t \in \mathbb{R}_+, \qquad S_0 > 0,$$

and the discounted asset price satisfies

$$dX_t = \sigma X_t d\tilde{B}_t, \qquad t \in \mathbb{R}_+, \qquad X_0 = S_0 > 0,$$

where $(\tilde{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$.

The following proposition applies to arbitrary square-integrable payoff functions, *i.e.* it covers exotic and path-dependent options.

Proposition 6.5. *Consider a random payoff $C \in L^2(\Omega)$ such that (6.8) holds, and let*

$$\xi_t = \frac{e^{-r(T-t)}}{\sigma S_t}\zeta_t, \tag{6.9}$$

$$\eta_t = \frac{e^{-r(T-t)}\,\mathbb{E}^*[C|\mathcal{F}_t] - \xi_t S_t}{A_t}, \qquad t \in [0,T]. \tag{6.10}$$

Then the portfolio $(\xi_t, \eta_t)_{t\in[0,T]}$ is self-financing, and letting

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \in [0,T], \tag{6.11}$$

we have

$$V_t = e^{-r(T-t)}\,\mathbb{E}^*[C|\mathcal{F}_t], \qquad t \in [0,T]. \tag{6.12}$$

In particular we have

$$V_T = C, \tag{6.13}$$

i.e. the portfolio $(\xi_t, \eta_t)_{t\in[0,T]}$ yields a hedging strategy leading to C, starting from the initial value

$$V_0 = e^{-rT}\,\mathbb{E}^*[C].$$

Proof. Relation (6.12) follows from (6.10) and (6.11), and it implies

$$V_0 = e^{-rT}\,\mathbb{E}^*[C] = \eta_0 A_0 + \xi_0 S_0$$

at $t = 0$, and (6.13) at $t = T$. It remains to show that the portfolio strategy $(\xi_t, \eta_t)_{t\in[0,T]}$ is self-financing. By (6.8) and Proposition 6.1 we have

$$\begin{aligned}
V_t = \eta_t A_t + \xi_t S_t &= e^{-r(T-t)}\,\mathbb{E}^*[C|\mathcal{F}_t] \\
&= e^{-r(T-t)}\,\mathbb{E}^*\left[\mathbb{E}^*[C] + \int_0^T \zeta_u d\tilde{B}_u \Big| \mathcal{F}_t\right] \\
&= e^{-r(T-t)}\left(\mathbb{E}^*[C] + \int_0^t \zeta_u d\tilde{B}_u\right) \\
&= e^{rt}V_0 + e^{-r(T-t)}\int_0^t \zeta_u d\tilde{B}_u \\
&= e^{rt}V_0 + \sigma\int_0^t \xi_u S_u e^{r(t-u)} d\tilde{B}_u \\
&= e^{rt}V_0 + \sigma\int_0^t \xi_u X_u e^{rt} d\tilde{B}_u \\
&= e^{rt}V_0 + e^{rt}\int_0^t \xi_u dX_u, \qquad t \in [0,T],
\end{aligned}$$

which shows that the discounted portfolio value $\widetilde{V}_t = e^{-rt}V_t$ satisfies

$$\widetilde{V}_t = V_0 + \int_0^t \xi_u dX_u, \qquad t \in [0,T],$$

and this implies that $(\xi_t, \eta_t)_{t\in[0,T]}$ is self-financing by Lemma 5.1. □

The above proposition shows that there always exists a hedging strategy starting from

$$V_0 = \mathbb{E}^*[C]e^{-rT}.$$

In addition, since there exists a hedging strategy leading to

$$\widetilde{V}_T = e^{-rT}C,$$

then $(\widetilde{V}_t)_{t\in[0,T]}$ is necessarily a martingale with

$$\widetilde{V}_t = \mathbb{E}^*\left[\widetilde{V}_T \Big| \mathcal{F}_t\right] = e^{-rT}\,\mathbb{E}^*[C|\mathcal{F}_t], \qquad t \in [0,T],$$

and initial value

$$\widetilde{V}_0 = \mathbb{E}^*\left[\widetilde{V}_T\right] = e^{-rT}\,\mathbb{E}^*[C].$$

In practice, the hedging problem can now be reduced to the computation of the process $(\zeta_t)_{t\in[0,T]}$ appearing in (6.8). This computation, called the Delta hedging, can be performed by application of the Itô formula and the Markov property, see e.g. [95]. Consider the Markov semi-group $(P_t)_{0\le t\le T}$ associated to $(S_t)_{t\in[0,T]}$, and defined by

$$P_t f(S_u) = \mathbb{E}^*[f(S_{t+u}) \mid \mathcal{F}_u] = \mathbb{E}^*[f(S_{t+u}) \mid S_u], \quad t, u \in \mathbb{R}_+,$$

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which acts on functions $f \in \mathcal{C}_b^2(\mathbb{R})$, with

$$P_t P_s = P_{s+t}, \qquad s,t \in \mathbb{R}_+.$$

Note that $(P_{T-t}f(S_t))_{t\in[0,T]}$ is an $\mathcal{F}_t$-martingale, *i.e.*:

$$\begin{aligned}\mathbb{E}^*[P_{T-t}f(S_t) \mid \mathcal{F}_u] &= \mathbb{E}^*[\mathbb{E}^*[f(S_T) \mid \mathcal{F}_t] \mid \mathcal{F}_u] \\ &= \mathbb{E}^*[f(S_T) \mid \mathcal{F}_u] \\ &= P_{T-u}f(S_u), \end{aligned} \tag{6.14}$$

$0 \leq u \leq t \leq T$, and we have

$$P_{t-u}f(x) = \mathbb{E}^*[f(S_t) \mid S_u = x] = \mathbb{E}^*[f(xS_t/S_u)], \qquad 0 \leq u \leq t. \tag{6.15}$$

The next lemma allows us to compute the process $(\zeta_t)_{t\in[0,T]}$ in case the payoff C is of the form $C = \phi(S_T)$ for some function ϕ. In case $C \in L^2(\Omega)$ is the payoff of an exotic option, the process $(\zeta_t)_{t\in[0,T]}$ can be computed using the Malliavin gradient on the Wiener space, cf. [82], [89].

Lemma 6.2. *Let $\phi \in \mathcal{C}_b^2(\mathbb{R}^n)$. The predictable representation*

$$\phi(S_T) = \mathbb{E}^*[\phi(S_T)] + \int_0^T \zeta_t d\tilde{B}_t \tag{6.16}$$

is given by

$$\zeta_t = \sigma S_t \frac{\partial}{\partial x}(P_{T-t}\phi)(S_t), \qquad t \in [0,T]. \tag{6.17}$$

Proof. Since $P_{T-t}\phi$ is in $\mathcal{C}^2(\mathbb{R})$, we can apply the Itô formula to the process

$$t \longmapsto P_{T-t}\phi(S_t) = \mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_t],$$

which is a martingale from the "tower property" (6.1) of conditional expectations as in (6.14). From the fact that the finite variation term in the Itô formula vanishes when $(P_{T-t}\phi(S_t))_{t\in[0,T]}$ is a martingale, (see e.g. Corollary II-6-1 page 72 of [96]), we obtain:

$$P_{T-t}\phi(S_t) = P_T\phi(S_0) + \sigma \int_0^t S_u \frac{\partial}{\partial x}(P_{T-u}\phi)(S_u)d\tilde{B}_u, \quad t \in [0,T], \tag{6.18}$$

with $P_T\phi(S_0) = \mathbb{E}^*[\phi(S_T)]$. Letting $t = T$, we obtain (6.17) by uniqueness of the predictable representation (6.16) of $C = \phi(S_T)$. □

By (6.15) we also have

$$\begin{aligned}\zeta_t &= \sigma S_t \frac{\partial}{\partial x} \mathbb{E}^*[\phi(S_T) \mid S_t = x]_{x=S_t} \\ &= \sigma S_t \frac{\partial}{\partial x} \mathbb{E}^*[\phi(xS_T/S_t)]_{x=S_t}, \qquad t \in [0,T],\end{aligned}$$

hence

$$\xi_t = \frac{1}{\sigma S_t} e^{-r(T-t)} \zeta_t \tag{6.19}$$
$$= e^{-r(T-t)} \frac{\partial}{\partial x} \mathbb{E}^*[\phi(xS_T/S_t)]_{x=S_t}, \qquad t \in [0,T],$$

which recovers the formula (5.10) for the Delta of a vanilla option. As a consequence we have $\xi_t \geq 0$ and there is no short selling when the payoff function ϕ is nondecreasing.

In the case of European options, the process ζ can be computed via the next proposition.

Proposition 6.6. *Assume that* $C = (S_T - K)^+$*. Then for* $0 \leq t \leq T$ *we have*

$$\zeta_t = \sigma S_t \, \mathbb{E}^* \left[\frac{S_T}{S_t} \mathbf{1}_{[K,\infty)} \left(x \frac{S_T}{S_t} \right) \right]_{x=S_t}.$$

Proof. This result follows from Lemma 6.2 and the relation $P_{T-t} f(x) = \mathbb{E}^*[f(S^x_{t,T})]$, after approximation of $x \longmapsto \phi(x) = (x-K)^+$ with $\mathcal{C}^2$ functions. □

From Proposition 6.6 we can recover the formula for the Delta of a European call option in the Black-Scholes model, cf. Proposition 5.5. Proposition 6.7 shows that the Black-Scholes self-financing hedging strategy is to hold a (possibly fractional) quantity

$$\xi_t = \Phi(d_+) = \Phi\left(\frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right) \geq 0 \tag{6.20}$$

of the risky asset, and to borrow a quantity

$$-\eta_t = K e^{-rT} \Phi\left(\frac{\log(S_t/K) + (r - \sigma_t^2/2)(T-t)}{\sigma\sqrt{T-t}} \right) \leq 0 \tag{6.21}$$

of the riskless (savings) account, cf. also Corollary 10.2 in Chapter 10.

In the next proposition we provide another proof of the result of Proposition 5.5.

Proposition 6.7. *The Delta of a European call option with payoff function* $f(x) = (x - K)^+$ *is given by*

$$\xi_t = \Phi(d_+) = \Phi\left(\frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right), \qquad 0 \leq t \leq T.$$

Proof. By Propositions 6.5 and 6.6 we have

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$$\begin{aligned}
\xi_t &= \frac{1}{\sigma S_t} e^{-r(T-t)} \zeta_t \\
&= e^{-r(T-t)} \, \mathbb{E}^* \left[\frac{S_T}{S_t} \mathbf{1}_{[K,\infty)} \left(x \frac{S_T}{S_t} \right) \right]_{x=S_t} \\
&= e^{-r(T-t)} \\
&\times \mathbb{E}^* \left[e^{\sigma(\tilde{B}_T - \tilde{B}_t) - \sigma^2(T-t)/2 + r(T-t)} \mathbf{1}_{[K,\infty)} \big(x e^{\sigma(\tilde{B}_T - \tilde{B}_t) - \sigma^2(T-t)/2 + r(T-t)} \big) \right]_{x=S_t} \\
&= \frac{1}{\sqrt{2\pi(T-t)}} \int_{\sigma(T-t)/2 - r(T-t)/\sigma + \sigma^{-1}\log(K/S_t)}^{\infty} e^{\sigma y - \sigma^2(T-t)/2 - y^2/(2(T-t))} dy \\
&= \frac{1}{\sqrt{2\pi(T-t)}} \int_{-d_-/\sqrt{T-t}}^{\infty} e^{-(y-\sigma(T-t))^2/(2(T-t))} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-d_-}^{\infty} e^{-(y-\sigma(T-t))^2/2} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-d_+}^{\infty} e^{-y^2/2} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_+} e^{-y^2/2} dy \\
&= \Phi(d_+).
\end{aligned}$$

□

As noted above, the result of Proposition 6.7 also follows from (5.10) or (6.19) and direct differentiation of the Black-Scholes function, cf. (5.16). In Figure 6.2 we plot the value of the Delta of a European as a function of the underlying and of time to maturity.

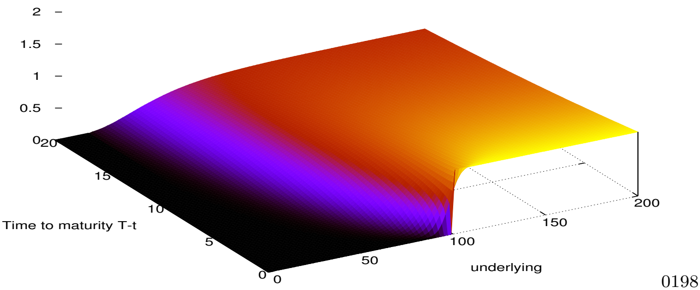

Fig. 6.2: Delta of a European option with strike $K = 100$.

The *gamma* of the European call option is defined as the second derivative of the option price with respect to the underlying, *i.e.*

$$\gamma_t = \frac{1}{S_t \sigma \sqrt{2\pi(T-t)}} \exp\left(-\frac{1}{2}\left(\frac{\log(S_t/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)\right)$$

In Figure 6.3 we plot the (truncated) value of the Gamma of a European as a function of the underlying and of time to maturity.

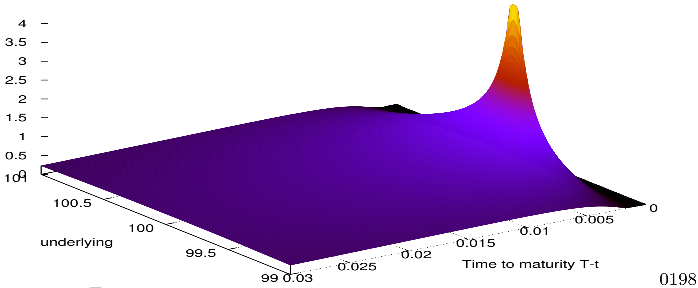

Fig. 6.3: Gamma of a European option with strike $K = 100$.

Since Gamma is always nonnegative, the Black-Scholes hedging strategy is to keep buying the underlying risky asset when its price increases, and to sell it when its price decreases, as can be checked from Figure 6.3.

Exercises

Exercise 6.1 Consider an asset price $(S_t)_{t\in\mathbb{R}_+}$ which is a martingale under the risk-neutral measure $\mathbb{P}^*$ in a market with interest rate $r = 0$, and let $\phi(x) = (x-K)^+$ be the (convex) European call payoff function.

Show that, for any two maturities $T_1 < T_2$ and $p, q \in [0,1]$ such that $p+q = 1$, the price of the average option with payoff $\phi(pS_{T_1} + qS_{T_2})$ is upper bounded by the price of the European call option with maturity T_2, *i.e.* show that

$$\mathbb{E}^*[\phi(pS_{T_1} + qS_{T_2})] \leq \mathbb{E}^*[\phi(S_{T_2})].$$

Hint 1: For ϕ a convex function we have $\phi(px+qy) \leq p\phi(x) + q\phi(y)$ for any $x, y \in \mathbb{R}$ and $p, q \in [0,1]$ such that $p+q = 1$.

Hint 2: Any convex function $\phi(S_t)$ of a martingale S_t is a *sub*martingale.

Exercise 6.2 Consider an underlying asset price process $(S_t)_{t\in\mathbb{R}_+}$.

a) Show that the price at time t of a European call option with strike price K and maturity T is lower bounded by $(S_t - Ke^{-r(T-t)})^+$, *i.e.*

$$e^{-r(T-t)}\,\mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t] \geq (S_t - Ke^{-r(T-t)})^+, \quad t \in [0, T].$$

b) Show that the price at time t of a European putoption with strike price K and maturity T is lower bounded by $Ke^{-r(T-t)} - S_t$, *i.e.*

$$e^{-r(T-t)}\,\mathbb{E}^*[(K - S_T)^+ \mid \mathcal{F}_t] \geq Ke^{-r(T-t)})^+ - S_t, \quad t \in [0, T].$$

Exercise 6.3 Forward start options [100]. Given two maturity dates $T_1 < T_2$, compute the price

$$e^{-r(T_1-t)}\,\mathbb{E}^*\left[e^{-r(T_2-T_1)}\,\mathbb{E}^*\left[(S_{T_2} - S_{T_1})^+ \mid \mathcal{F}_{T_1}\right] \mid \mathcal{F}_t\right]$$

at time $t \in [0, T_1]$, of a *forward start* European call option, *i.e.* an option whose holder receives at time T_1 the value of a standard European call option at the money, with maturity T_2.

Exercise 6.4 Consider the price process $(S_t)_{t\in[0,T]}$ given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t$$

and a riskless asset of value $A_t = A_0 e^{rt}$, $t \in [0, T]$, with $r > 0$. In this problem, $(\eta_t, \xi_t)_{t\in[0,T]}$ denotes a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad 0 \leq t \leq T.$$

a) Compute the arbitrage price

$$C(t, S_t) = e^{-r(T-t)}\,\mathbb{E}^*[|S_T|^2 \mid \mathcal{F}_t],$$

at time $t \in [0, T]$, of the power option with payoff $|S_T|^2$.

b) Compute a self-financing portfolio strategy $(\eta_t, \xi_t)_{t\in[0,T]}$ hedging the claim $|S_T|^2$.

Exercise 6.5 Let again $(\eta_t, \xi_t)_{t\in[0,T]}$ denote a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad 0 \leq t \leq T,$$

where S_t, *resp.* A_t, denotes the price at time t of a risky, *resp.* riskless, asset.

a) Solve the stochastic differential equation

$$dS_t = \alpha S_t dt + \sigma dB_t$$

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in terms of $\alpha, \sigma > 0$, and the initial condition S_0.

b) For which value α_M of α is the discounted price process $\tilde{S}_t = e^{-rt} S_t$, $t \in [0, T]$, a martingale under $\mathbb{P}$?

c) For each value of α, build a probability measure $\mathbb{P}_\alpha$ under which the discounted price process $\tilde{S}_t = e^{-rt} S_t$, $t \in [0, T]$, is a martingale.

d) Compute the arbitrage price

$$C(t, S_t) = e^{-r(T-t)} \mathbb{E}_\alpha[\exp(S_T) \mid \mathcal{F}_t]$$

at time $t \in [0, T]$ of the contingent claim with payoff $\exp(S_T)$, and recover the result of Exercise 5.1.

e) Explicitly compute the portfolio strategy $(\eta_t, \xi_t)_{t \in [0,T]}$ that hedges the contingent claim $\exp(S_T)$.

f) Check that this strategy is self-financing.

Exercise 6.6 Let $(B_t)_{t \in \mathbb{R}_+}$ be a standard Brownian motion generating a filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$. Recall that for $f \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R})$, Itô's formula for Brownian motion reads

$$f(t, B_t) = f(0, B_0) + \int_0^t \frac{\partial f}{\partial s}(s, B_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s) dB_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s) ds.$$

a) Let $r \in \mathbb{R}$, $\sigma > 0$, $f(x, t) = e^{rt + \sigma x - \sigma^2 t/2}$, and $S_t = f(t, B_t)$. Compute $df(t, B_t)$ by Itô's formula, and show that S_t solves the stochastic differential equation

$$dS_t = r S_t dt + \sigma S_t dB_t,$$

where $r > 0$ and $\sigma > 0$.

b) Show that

$$\mathbb{E}\left[e^{\sigma B_T} \mid \mathcal{F}_t\right] = e^{\sigma B_t + \sigma^2 (T-t)/2}, \qquad 0 \leq t \leq T.$$

Hint: Use the independence of increments in the decomposition

$$B_T = (B_T - B_t) + (B_t - B_0)$$

and the Laplace transform $\mathbb{E}[e^{\alpha X}] = e^{\alpha^2 \eta^2 / 2}$ when $X \simeq \mathcal{N}(0, \eta^2)$.

c) Show that the process $(S_t)_{t \in \mathbb{R}_+}$ satisfies

$$\mathbb{E}[S_T \mid \mathcal{F}_t] = e^{r(T-t)} S_t, \qquad 0 \leq t \leq T.$$

d) Let $C = S_T - K$ denote the payoff of a forward contract with exercise price K and maturity T. Compute the discounted expected payoff

$$V_t := e^{-r(T-t)} \mathbb{E}[C \mid \mathcal{F}_t].$$

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e) Find a self-financing portfolio strategy $(\xi_t, \eta_t)_{t\in\mathbb{R}_+}$ such that

$$V_t = \xi_t S_t + \eta_t A_t, \qquad 0 \le t \le T,$$

where $A_t = A_0 e^{rt}$ is the price of a riskless asset with interest rate $r > 0$. Show that it recovers the result of Exercise 5.3-(c).

f) Show that the portfolio $(\xi_t, \eta_t)_{t\in[0,T]}$ found in Question (e) *hedges* the payoff $C = S_T - K$ at time T, *i.e.* show that $V_T = C$.

Exercise 6.7 Digital options. Consider a price process $(S_t)_{t\in\mathbb{R}_+}$ given by

$$\frac{dS_t}{S_t} = r dt + \sigma dB_t, \qquad S_0 = 1,$$

under the risk-neutral measure $\mathbb{P}$. A digital (or binary) *call*, *resp. put*, option is a contract with maturity T, strike K, and payoff

$$C_d := \begin{cases} \$1 & \text{if } S_T \ge K, \\ 0 & \text{if } S_T < K, \end{cases} \qquad \textit{resp.} \quad P_d := \begin{cases} \$1 & \text{if } S_T \le K, \\ 0 & \text{if } S_T > K. \end{cases}$$

Recall that the prices $\pi_t(C_d)$ and $\pi_t(P_d)$ at time t of the digital call and put options are given by the discounted expected payoffs

$$\pi_t(C_d) = e^{-r(T-t)}\,\mathbb{E}[C_d \mid \mathcal{F}_t] \quad \text{and} \quad \pi_t(P_d) = e^{-r(T-t)}\,\mathbb{E}[P_d \mid \mathcal{F}_t]. \qquad (6.22)$$

a) Show that the payoffs C_d and P_d can be rewritten as

$$C_d = \mathbf{1}_{[K,\infty)}(S_T) \qquad \text{and} \qquad P_d = \mathbf{1}_{[0,K]}(S_T).$$

b) Using Relation (6.22), Question (a), and the relation

$$\mathbb{E}[\mathbf{1}_{[K,\infty)}(S_T) \mid S_t = x] = \mathbb{P}(S_T \ge K \mid S_t = x),$$

show that the price $\pi_t(C_d)$ is given by

$$\pi_t(C_d) = C_d(t, S_t),$$

where $C_d(t,x)$ is the function defined by

$$C_d(t,x) := e^{-r(T-t)}\mathbb{P}(S_T \ge K \mid S_t = x).$$

c) Using the results of Exercise 4.10-(c) and of Question (b), show that the price $\pi_t(C_d)$ of the digital call option is given by

$$C_d(t,x) = e^{-r(T-t)}\Phi\left(\frac{(r-\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right)$$

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$$= e^{-r(T-t)}\Phi(d_-),$$

where

$$d_- = \frac{(r - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma\sqrt{T-t}}.$$

d) Assume that the binary option holder is entitled to receive a "return amount" $\alpha \in [0,1]$ in case the underlying ends out of the money at maturity. Compute price at time $t \in [0,T]$ of this modified contract.

e) Using Relation (6.22) and Question (a), prove the call-put parity relation

$$\pi_t(C_d) + \pi_t(P_d) = e^{-r(T-t)}, \qquad 0 \le t \le T. \qquad (6.23)$$

If needed, you may use the fact that $\mathbb{P}(S_T = K) = 0$.

f) Using the results of Questions (e) and (c), show that the price $\pi_t(P_d)$ of the digital put is given by

$$\pi_t(P_d) = e^{-r(T-t)}\Phi(-d_-).$$

g) Using the result of Question (c), compute the Delta

$$\xi_t := \frac{\partial C_d}{\partial x}(t, S_t)$$

of the digital call option. Does the Black-Scholes hedging strategy of such a call option involve short-selling ? Why ?

h) Using the result of Question (f), compute the Delta

$$\xi_t := \frac{\partial P_d}{\partial x}(t, S_t)$$

of the digital put option. Does the Black-Scholes hedging strategy of such a put option involve short-selling ? Why ?

Exercise 6.8 Option pricing with dividends. (Exercise 5.7 continued) Consider an underlying asset price process $(S_t)_{t\in\mathbb{R}_+}$ modeled under the risk-neutral measure as

$$dS_t = (r - D)S_t dt + \sigma S_t dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion and $D > 0$ is a continuous-time dividend rate. Compute the price at time $t \in [0,T]$ of the European call option in a market with dividend rate D by the martingale method.

Exercise 6.9 Log options.

a) Consider a market model made of a risky asset with price $(S_t)_{t\in\mathbb{R}_+}$ as in Exercise 4.12-(d) and a riskless asset with price $A_t = \$1 \times e^{rt}$ and riskless interest rate $r = \sigma^2/2$. From the answer to Exercise 4.12-(b), show that the arbitrage price

$$V_t = e^{-r(T-t)} \mathbb{E}[(\log S_T)^+ \mid \mathcal{F}_t]$$

at time $t \in [0, T]$ of a log call option with payoff $(\log S_T)^+$ is equal to

$$V_t = \sigma e^{-r(T-t)} \sqrt{\frac{T-t}{2\pi}} e^{-B_t^2/(2(T-t))} + \sigma e^{-r(T-t)} B_t \Phi\left(\frac{B_t}{\sqrt{T-t}}\right).$$

b) Show that V_t can be written as

$$V_t = g(T-t, S_t),$$

where $g(\tau, x) = e^{-r\tau} f(\tau, \log x)$, and

$$f(\tau, y) = \sigma \sqrt{\frac{\tau}{2\pi}} e^{-y^2/(2\sigma^2\tau)} + y\Phi\left(\frac{y}{\sigma\sqrt{\tau}}\right).$$

c) Figure 6.4 represents the graph of $(\tau, x) \mapsto g(\tau, x)$, with $r = 0.05 = 5\%$ *per year* and $\sigma = 0.1$. Assume that the current underlying price is \$1 and there remains 700 days to maturity. What is the price of the option ?

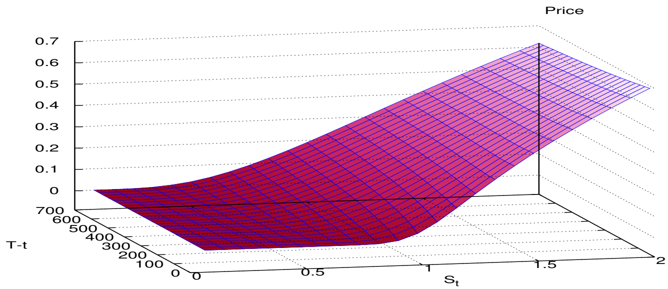

Fig. 6.4: Option price as a function of the underlying and of time to maturity

d) Show* that the (possibly fractional) quantity $\xi_t = \frac{\partial g}{\partial x}(T-t, S_t)$ of S_t at time t in a portfolio hedging the payoff $(\log S_T)^+$ is equal to

$$\xi_t = e^{-r(T-t)} \frac{1}{S_t} \Phi\left(\frac{\log S_t}{\sigma\sqrt{T-t}}\right), \qquad 0 \leq t \leq T.$$

e) Figure 6.5 represents the graph of $(\tau, x) \mapsto \frac{\partial g}{\partial x}(\tau, x)$. Assuming that the current underlying price is \$1 and that there remains 700 days to maturity,

* Recall the chain rule of derivation $\frac{\partial}{\partial x} f(\tau, \log x) = \frac{1}{x} \frac{\partial f}{\partial y}(\tau, y)_{|y=\log x}$.

how much of the risky asset should you hold in your portfolio in order to hedge one log option ?

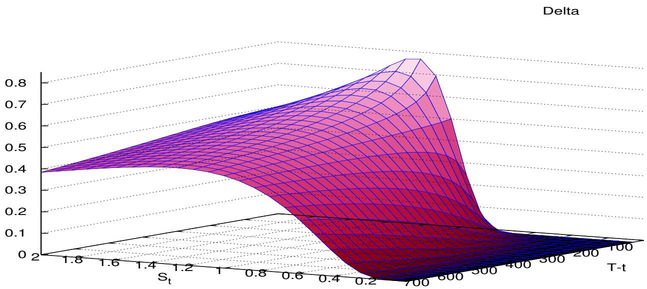

Fig. 6.5: Delta as a function of the underlying and of time to maturity

f) Based on the framework and answers of Questions (c) and (e), should you borrow or lend the riskless asset $A_t = \$1 \times e^{rt}$, and for what amount ?

g) Show that the Gamma of the portfolio, defined as $\Gamma_t = \frac{\partial^2 g}{\partial x^2}(T-t, S_t)$, equals

$$\Gamma_t = e^{-r(T-t)} \frac{1}{S_t^2} \left(\frac{1}{\sigma\sqrt{2\pi(T-t)}} e^{-(\log S_t)^2/(2\sigma^2(T-t))} - \Phi\left(\frac{\log S_t}{\sigma\sqrt{T-t}} \right) \right),$$

$0 \leq t \leq T$.

h) Figure 6.6 represents the graph of Gamma. Assume that there remains 60 days to maturity and that S_t, currently at \$1, is expected to increase. Should you buy or (short) sell the underlying asset in order to hedge the option ?

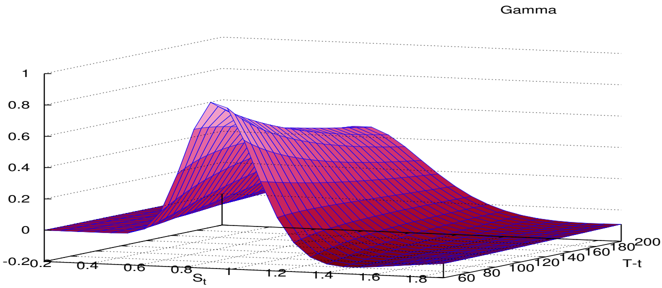

Fig. 6.6: Gamma as a function of the underlying and of time to maturity

i) Let now $\sigma = 1$. Show that the function $f(\tau, y)$ of Question (b) solves the *heat equation*

$$\begin{cases} \frac{\partial f}{\partial \tau}(\tau, y) = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\tau, y) \\ \\ f(0, y) = (y)^+. \end{cases}$$

Exercise 6.10 Log options with given strike.

a) Consider a market model made of a risky asset with price $(S_t)_{t\in\mathbb{R}_+}$ as in Exercise 4.10, a riskless asset with price $A_t = \$1 \times e^{rt}$, riskless interest rate $r = \sigma^2/2$ and $S_0 = 1$. From the answer to Exercise 16.4-(b), show that the arbitrage price

$$V_t = e^{-r(T-t)}\,\mathbb{E}^*[(K - \log S_T)^+ \mid \mathcal{F}_t]$$

at time $t \in [0, T]$ of a log call option with strike K and payoff $(K - \log S_T)^+$ is equal to

$$V_t = \sigma e^{-r(T-t)}\sqrt{\frac{T-t}{2\pi}}e^{-(B_t - K/\sigma)^2/(2(T-t))} + e^{-r(T-t)}(K - \sigma B_t)\Phi\left(\frac{K/\sigma - B_t}{\sqrt{T-t}}\right).$$

b) Show that V_t can be written as

$$V_t = g(T - t, S_t),$$

where $g(\tau, x) = e^{-r\tau}f(\tau, \log x)$, and

$$f(\tau, y) = \sigma\sqrt{\frac{\tau}{2\pi}}e^{-(K-y)^2/(2\sigma^2\tau)} + (K - y)\Phi\left(\frac{K - y}{\sigma\sqrt{\tau}}\right).$$

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c) Figure 6.7 represents the graph of $(\tau, x) \mapsto g(\tau, x)$, with $r = 0.125$ *per year* and $\sigma = 0.5$. Assume that the current underlying price is \$3 and there remains 700 days to maturity. What is the price of the option ?

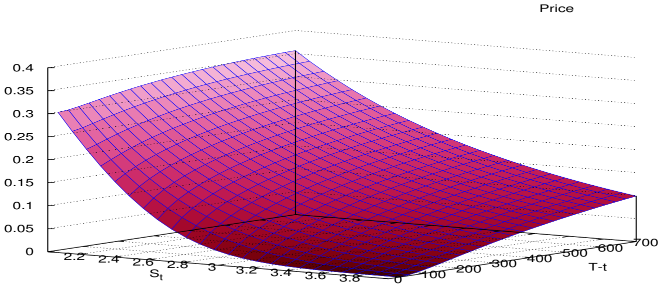

Fig. 6.7: Option price as a function of the underlying and of time to maturity

d) Show* that the quantity $\xi_t = \frac{\partial g}{\partial x}(T-t, S_t)$ of S_t at time t in a portfolio hedging the payoff $(K - \log S_T)^+$ is equal to

$$\xi_t = -e^{-r(T-t)} \frac{1}{S_t} \Phi\left(\frac{K - \log S_t}{\sigma\sqrt{T-t}}\right), \qquad 0 \leq t \leq T.$$

e) Figure 6.8 represents the graph of $(\tau, x) \mapsto \frac{\partial g}{\partial x}(\tau, x)$. Assuming that the current underlying price is \$3 and that there remains 700 days to maturity, how much of the risky asset should you hold in your portfolio in order to hedge one log option ?

* Recall the chain rule of derivation $\frac{\partial}{\partial x} f(\tau, \log x) = \frac{1}{x}\frac{\partial f}{\partial y}(\tau, y)_{|y=\log x}$.

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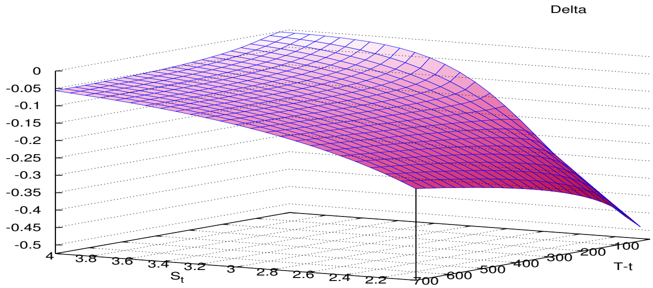

Fig. 6.8: Delta as a function of the underlying and of time to maturity

f) Based on the framework and answers of Questions (c) and (e), should you borrow or lend the riskless asset $A_t = \$1 \times e^{rt}$, and for what amount ?

g) Show that the Gamma of the portfolio, defined as $\Gamma_t = \frac{\partial^2 g}{\partial x^2}(T-t, S_t)$, equals

$$\Gamma_t = e^{-r(T-t)} \frac{1}{S_t^2} \left(\frac{1}{\sigma\sqrt{2\pi(T-t)}} e^{-(K-\log S_t)^2/(2\sigma^2(T-t))} + \Phi\left(\frac{K-\log S_t}{\sigma\sqrt{T-t}}\right)\right), \qquad 0 \leq t \leq$$

h) Figure 6.9 represents the graph of Gamma. Assume that there remains 10 days to maturity and that S_t, currently at \$3, is expected to increase. Should you buy or (short) sell the underlying asset in order to hedge the option ?

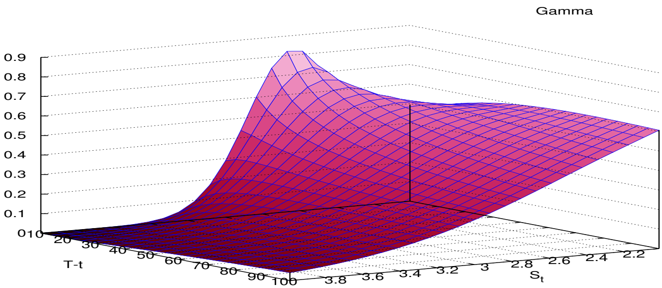

Fig. 6.9: Gamma as a function of the underlying and of time to maturity

i) Show that the function $f(\tau, y)$ of Question (b) solves the *heat equation*

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$$\begin{cases} \dfrac{\partial f}{\partial \tau}(\tau, y) = \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 f}{\partial y^2}(\tau, y) \\ \\ f(0, y) = (K - y)^+. \end{cases}$$