Example

$U(x) = x^p$

$V_N(x) = U(x)$

$$V_{N-1}(x) = \sup_\alpha \mathbb{E}[U(X_N)]$$

$$= \sup_\alpha \mathbb{E}\left[(\alpha z x + (1-\alpha)(1+r)x)^p\right]$$

$$= x^p \underbrace{\sup_\alpha \mathbb{E}\left[(\alpha z + (1-\alpha)(1+r))^p\right]}_{V_{N-1}(1)}$$

$$V_{N-1}(x) = x^p K$$

DPP

$$V_{N-2}(x) = \sup_\alpha \mathbb{E}\left[V_{N-1}(X_{N-1})\right]$$

$$= \sup_\alpha \mathbb{E}\left[(X_{N-1})^p K\right]$$

$$= x^p K \underbrace{\sup_\alpha \mathbb{E}\left[\alpha z + (1-\alpha)(1+r)\right]^p}$$

$$= x^p K^2$$

$$V_{N-i} = x^p K^i$$

Induction suppose it's true for $N-i$ check $N-(i+1)$

$$V_{N-(i+1)}(x) = \sup_\alpha \mathbb{E}\left[V_{N-i}(X_{N-i})\right]$$

$$= \sup_\alpha \mathbb{E}\left[(X_{N-i})^p K^i\right]$$

$$= K^i x^p \sup_\alpha \mathbb{E}\left[\alpha z + (1-\alpha)(1+r)\right]^p$$

$$= K^{i+1} x^p \quad \text{true.}$$

$$U(x) = \ln(x)$$

$$V_N(x) = U(x)$$

$$V_{N-1}(x) = \sup_{\alpha} \mathbb{E}\left[U(X_N) \right]$$

$$= \sup_{\alpha} \mathbb{E}\left[\ln(X_N) \right] = \sup_{\alpha} \mathbb{E}\left[\ln\left(x(\alpha \zeta + (1-\alpha)(1+r)) \right) \right]$$

$$\ln(a \cdot b) = \ln a + \ln b$$

$$= \sup_{\alpha} \mathbb{E}\left[\ln(x) + \ln\left(\alpha\zeta + (1-\alpha)(1+r) \right) \right]$$

$$= \ln(x) + \underbrace{\sup_{\alpha} \mathbb{E}\left[\ln\left(\alpha\zeta + (1-\alpha)(1+r) \right) \right]}_{V_{N-1}(1)}$$

$$V_{N-1}(x) = \ln(x) + V_{N-1}(1)$$

DPP

$$V_{N-2}(x) = \sup_{\alpha} \mathbb{E}\left[V_{N-1}(X_{N-1}) \right]$$

$$= \sup_{\alpha} \mathbb{E}\left[\ln(X_{N-1}) + V_{N-1}(1) \right]$$

$$= \sup_{\alpha} \mathbb{E}\left[\ln(x) + \ln\left(\alpha\zeta + (1-\alpha)(1+r) \right) + V_{N-1}(1) \right]$$

$$= \ln(x) + \sup_{\alpha} \mathbb{E}\left[\ln\left(\alpha\zeta + (1-\alpha)(1+r) \right) \right] + V_{N-1}(1)$$

$$= \ln(x) + 2 V_{N-1}(1)$$

propose $V_{N-K}(x) = \ln(x) + K V_{N-1}(1)$

then for $K+1$

$$V_{N-(K+1)}(x) = \sup_{\alpha} \mathbb{E}\left[V_{N-K}(X_{N-K}) \right]$$

$$= \sup_{\alpha} \mathbb{E}\left[\ln(X_{N-K}) + K V_{N-1}(1) \right]$$

$$= K V_{N-1}(1) + \sup_{\alpha} \mathbb{E}\left[\ln(X_{N-K}) \right] = \ln(x) + (K+1) V_{N-1}(1)$$

$$U(x) = x^p$$
$$V_N(x) = U(x)$$

$$V_{N-1}(x) = x^p V_{N-1}(1)$$

$$V_{N-2}(x) = \sup_{\alpha, c} \mathbb{E}\left[U(c X_{N-1}) + V_{N-1}((1-c) X_{N-1}) \right]$$

$$= \sup_{\alpha, c} \mathbb{E}\left[x^p c^p (\alpha \zeta + (1-\alpha)(1+r))^p + (1-c)^p (X_{N-1})^p V_{N-1}(1) \right]$$

$$= x^p \underbrace{\sup_{\alpha, c} \mathbb{E}\left[\left[c^p + (1-c)^p V_{N-1}(1)\right] \left[\alpha \zeta + (1-\alpha)(1+r)\right]^p \right]}_{V_{N-1}(c^p + (1-c)^p V_{N-1}(1))^{1/p}}$$

$$V_{N-3}(x) = \sup_{\alpha, c} \mathbb{E}\left[U(c X_{N-2}) + V_{N-2}((1-c) X_{N-2}) \right]$$

$$= \sup_{\alpha, c} \mathbb{E}\Big[c^p x^p (\alpha \zeta + (1-\alpha)(1+r))^p$$
$$+ (1-c)^p (X_{N-2})^p V_{N-1}(c^p + (1-c)^p V_{N-1}(1))^{1/p} \Big]$$

$$= x^p \underbrace{\sup_{\alpha, c} \mathbb{E}\left[\left(c^p + (1-c)^p V_{N-1}(c^p + (1-c)^p V_{N-1}(1))^{1/p}\right) \left(\alpha \zeta + (1-\alpha)(1+ \right.\right.}_{V_{N-1}\left[c^p + (1-c)^p V_{N-1}(c^p + (1-c)^p V_{N-1}(1))^{1/p}\right]^{1/p}}$$

$$K_N = V_{N-1}(1)$$
$$K_{N-1} = V_{N-1}(c^p + (1-c)^p K_N)^{1/p} \quad \Rightarrow$$
$$K_{N-2} = V_{N-1}(c^p + (1-c)^p K_{N-1})^{1/p}$$

$U(x) = \ln(x)$

$$V_{N-1}(x) = \sup \mathbb{E}\left[U(X_N)\right]$$
$$= \ln(x) + V_{N-1}(1)$$
$$= \ln(x) + C_{N-1}$$

$$V_{N-2}(x) = \sup_{\alpha, c} \mathbb{E}\left[U(cX_{N-1}) + V_{N-1}\left((1-c)X_{N-1}\right)\right]$$
$$= \sup_{\alpha, c} \mathbb{E}\left[\ln(cX_{N-1}) + \ln\left((1-c)X_{N-1}\right) + C_{N-1}\right]$$
$$= 2\ln(x) + \sup_{\alpha, c} \mathbb{E}\left[\ln\left(c(1-c)\right) + \ln\left(\alpha z + (1-\alpha)(1+r)\right)^2\right] + C_{N-1}$$

$$2\ln(x) + C_{N-1} + \underbrace{\underbrace{\sup_{\alpha, c} \mathbb{E}\left[\ln\left(c(1-c)\left(\alpha z + (1-\alpha)(1+r)\right)^2\right)\right]}_{V_{N-1}\left(c(1-c)\left[\alpha z + (1-\alpha)(1+r)\right]\right)}}_{C_{N-2}}$$

$$V_{N-3}(x) = \sup_{\alpha, c} \mathbb{E}\left[U(cX_{N-2}) + V_{N-2}\left(X_{N-2}(1-c)\right)\right]$$
$$= \sup_{\alpha, c} \mathbb{E}\left[\ln x + \ln\left(c(\alpha z + (1-\alpha)(1+r)\right) + 2\ln\left((1-c)X_{N-2}\right) + C_{N-1} + C_{N-2}\right]$$

$$= 3\ln(x) + \underbrace{\sup_{\alpha, c} \mathbb{E}\left[\ln\left(c(\alpha z + (1-\alpha)(1+r)\right) + 2\ln\left((1-c)(\alpha z + (1-\alpha)(1+r)\right)\right]}_{} + C_{N-1} + C_{N-2}$$

$$C_{N-3} = \sup_{\alpha, c} \mathbb{E}\left[\ln\left(c(1-c)^2\left(\alpha z + (1-\alpha)(1+r)\right)^3\right)\right]$$
$$C_{N-3} = V_{N-1}\left(c(1-c)^2(\alpha z + (1-\alpha)(1+r))^3\right)$$

$$J(t,x) = \sup_\alpha \mathbb{E}\left[g(X_T) + \int_t^T f(X_s)\,ds \right]$$

$f := 0 \quad g = x^p \qquad dX_t = [(\rho-\mu)\alpha + \mu] X_s\,ds + \sigma\alpha\,dB_s$

GBM $\quad X_T = x \exp\left\{ \int_t^T (\rho-\mu)\alpha + \mu - \frac{1}{2}\sigma^2\alpha^2\,ds + \int_t^T \sigma\alpha\,dB_s \right\}$

$$V(t,x) = \sup_\alpha \mathbb{E}\left[x^p \exp\left\{ p\int_t^T \quad ds + p\int_t^T \quad dB_s \right\} \right]$$

$$= x^p \sup_\alpha e^{p\int_t^T \quad ds} \underbrace{\mathbb{E}\left[e^{p\int_t^T \quad dB_s} \right]}_{\text{momentum function}} \qquad e^{\lambda m + \frac{1}{2}\lambda^2\sigma^2}$$

$$N\left(0, \int_t^T p^2\sigma^2\alpha^2\,ds\right)$$

$$= x^p \sup_\alpha e^{\int_t^T p[(\rho-\mu)\alpha + \mu - \frac{1}{2}\sigma^2\alpha^2]\,ds + \frac{p^2}{2}\int_t^T \sigma^2\alpha^2\,ds}$$

$$V(t,x) = x^p \sup_\alpha \underbrace{\exp\left\{ \int_t^T p[(\rho-\mu)\alpha + \mu + p(p-1)\frac{\sigma^2}{2}\alpha^2]\,ds \right\}}$$

$$C_{t,T} = \exp\left\{ \sup_\alpha \int_t^T \quad ds \right\}$$

$$C_{t,T} = C_{t,t+h} \cdot C_{t+h,T}$$

$$\dot{C}_{t,T} = \dot{C}_{t,t+h} \cdot C_{t+h,T} + C_{t,t+h} \cdot \dot{C}_{t+h,T}$$

$$\sup_{t,T} \int_t^T ds = \sup_{t,t+h} \int_t^{t+h} + \sup_{t+h,T} \int_{t+h}^T$$

because they are constan

$$\frac{d}{dt} \sup_\alpha \int_t^{t+h} f(s)\,ds =$$

$$\frac{d}{dt} \sup_\alpha \left(F(t+h) - F(t) \right) =$$

$$\frac{d}{dt} F_{t+h}(\alpha^*) - F_t(\alpha^*) =$$

$$f_{\alpha^*}(t+h) - f_{\alpha^*}(t)$$

$$\dot{C}_{t,T} = \left[f_{\alpha^*}(t+h) - f_{\alpha^*}(t) \right] C_{t+h,T} \cdot C_{t,t+h} + C_{t,t+h}\left[-f_{\alpha^*}(t+h) \right] C_{t+h,T}$$

$$\Rightarrow \dot{C}_{t,T} = -\sup f(t)\, C_{t,T}$$

for x^i, bernoulli, no consumption α is constant.

By Induction

$$X^*_{k+1} = \alpha^*_0 X^*_k \xi_{k+1} + (1-\alpha^*_0)(1+r)X^*_k$$

$$A = \left[0, \frac{W_0}{S_0}\right] \qquad \alpha \xi_1 x + (1-\alpha)(1+r)x = X_1$$

$$\xi = \left[c, \frac{1}{c}\right] \qquad a = \min\left[c, 1+r\right]$$

$$b = \max\left[\frac{1}{c}, 1+r\right] \qquad \xi \begin{cases} u \\ 1+r \\ d \end{cases}$$

$$a\,\alpha x + (1-\alpha)x\,a \le X_1 \le b\,\alpha x + (1-\alpha)x\,b$$

$$0 < a X_0 \le X_1 \le b X_0 < \infty$$

$$0 < X_1 < \infty$$

U is nondecreasing concave $\qquad U(0) < U(X_1) < U(bX_0) < \infty$

$$0 < |U X_1| < \infty \qquad \mathbb{E}\,U(X_1) < \infty$$

$$\sup_{\alpha} \mathbb{E}\,|U(X_1)| < \infty \quad \text{finite Expectati[on]}$$

Continuity $\alpha \in A$ is $J(\alpha)$ is cont $\Rightarrow \exists \sup J(\alpha)$

$$\alpha_n \longrightarrow \alpha$$

Then $U(\alpha_n \xi x + (1-\alpha_n)(1+r)x) \longrightarrow U(X_1)$

for $\mathbb{E}\,U(X_{\alpha_n})$ we need Dominated Convergence.

as $|U(X_{\alpha_n})|$ is bounded by $U(b_n X_0) < \infty$

limit inside

$$\mathbb{E}\,U(X_{\alpha_n}) \longrightarrow \mathbb{E}\,U(X_\alpha).$$

1 Period, Bernoulli, x^p.

$$\cancel{V_0(x) = \sup \mathbb{E}[x\alpha z + (1-\alpha)(1+r)x]^p}$$

$$V_K(x) = C^{N-K} x^p$$

$$V_1(x) = C^{N-1} x^p.$$

$$V_0(x) = C^N x^p$$

$$C = \sup_a \mathbb{E}\left[\alpha z + (1-\alpha)(1+r)\right]^p$$

find α. $\Rightarrow$ get C $\Rightarrow$ V_K.

Uniqueness

if $z = 1+r$ $\quad \alpha(1+r) + (1-\alpha)(1+r) = 1+r$.
so constant function of α infinite solutions

lets say $z \neq 1+r$. or $\mathbb{P}(z = 1+r) < 1$

Check

$\mathbb{P}(u) = \frac{1}{2}$

$$\mathbb{E}\left[\alpha z + (1-\alpha)(1+r)\right]^p = \frac{1}{2}(\alpha u + (1-\alpha)(1+r))^p + \frac{1}{2}(\alpha d + (1-\alpha)(1+r))^p.$$

$$\frac{\partial \mathbb{E}[\ ]^p}{\partial \alpha} = \frac{p}{2}(\ u\ )^{p-1}(u - (1+r)) + \frac{p}{2}(\ d\ )^{p-1}(d - (1+r))$$

$$\frac{\partial^2 \mathbb{E}[\ ]^p}{\partial \alpha^2} = \frac{p(p-1)}{2}(\ u\ )^{p-2}(u - (1+r))^2 + \frac{p(p-1)}{2}(\ d\ )^{p-2}(d - (1+r))^2 < 0$$

concave.

$p(p-1) < 0$

$(u-(1+r))^2 > 0$

$u^{p-2} > 0$

$\therefore$ α No Dependence on x or C

α is same each period.

Relax $A = [0,1]$

Keeping $X_1 \geq 0$

$$\begin{cases} \alpha u + (1-\alpha)(1+r) \geq 0 \\ \alpha d + (1-\alpha)(1+r) \geq 0 \end{cases}$$

$\forall \alpha > 0$, $\alpha < 0$

$\alpha(u-(1+r)) + 1+r \geq 0$

$\alpha(d-(1+r)) + 1+r \geq 0$

$\forall \alpha < 0$, $\alpha > 0$

$\alpha u + (1+r) - \alpha(1+r) \geq 0$

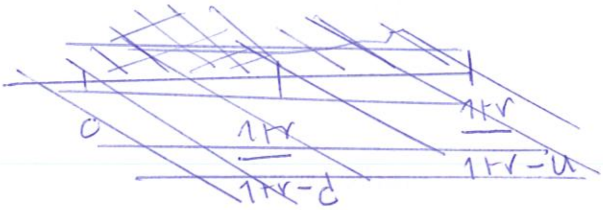
$\alpha(u-(1+r)) \geq -(1+r)$

$\alpha \leq \frac{1+r}{(1+r)-u}$ ϕ

$d < 1+r < u$

$1+r-u < 0$ neg

$0 < 1+r-d$ pos



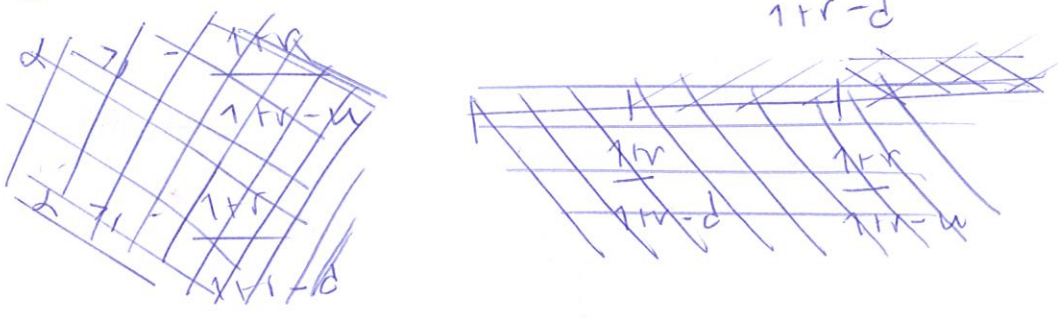
$\alpha \leq \frac{1+r}{1+r-d}$ *

α negative.

$\alpha \geq -\frac{1+r}{1+r-u}$ contradiction ϕ

~~if α is negative~~

$\alpha \geq -\frac{1+r}{1+r-d}$



$$A = \left[+\frac{1+r}{1+r-u}, \frac{1+r}{1+r-d} \right]$$

$\frac{1+r}{1+r-a}$ $\frac{1+r}{1+r-d}$

$$U(x) = -e^{-\lambda x}$$

$$V_{N-1}(x) = \sup_{\alpha} \mathbb{E}\left[U(X_N) \right]$$

$$\sup_{\alpha} \mathbb{E}\left[-e^{-\lambda(\alpha \xi x + (1-\alpha)(1+r)x)} \right]$$

$$\sup_{\alpha} \mathbb{E}\left[-e^{-\lambda(\alpha \xi x - \alpha x(1+r))} e^{-\lambda x(1+r)} \right]$$

$$V_{N-1}(x) = e^{-\lambda x(1+r)} \sup_{\alpha} \mathbb{E}\left[-e^{-\lambda x \alpha(\xi - (1+r))} \right]$$

$$V_{N-2}(x) = \sup_{\alpha} \mathbb{E}\left[V_{N-1}(X_{N-1}) \right]$$

$$= \sup_{\alpha} \mathbb{E}\left[C_{N-1} e^{-\lambda X_{N-1}(1+r)} \right]$$

$$= \sup_{\alpha} \left[C_{N-1} e^{-\lambda(1+r)(\alpha x \xi + (1-\alpha)(1+r)x)} \right]$$

$$(\alpha x \xi - \alpha(1+r)x) + (1+r)x$$

$$= e^{-\lambda(1+r)^2 x} C_{N-1} \sup_{\alpha} \mathbb{E}\left[e^{-\lambda(1+r)\alpha x(\xi - (1+r))} \right]$$

Merton Allocation

$$dX_t = \alpha_t \frac{X_t}{S_t} dS_t + (1-\alpha_t)\frac{X_t}{S^0_t} dS^0_t$$

$$dS_t = \mu S_t dt + \sigma S_t dB_t \qquad dS^0_t = r S^0_t dt$$

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

$$\frac{dX_t}{X_t} = \alpha_t \cdot (\mu dt + \sigma dB_t) + (1-\alpha_t) r dt$$

$$\frac{dX_t}{X_t} = [\alpha\mu + (1-\alpha) r] dt + \alpha \sigma dB_t$$

$$d\log(x) = \cancel{f_1 dt} + f_2 dx + \frac{1}{2} f_{22} dx^2 + \cancel{\frac{1}{2} f_{11} dt^2} + \cancel{\frac{1}{2} 2 f_{12} dt dx}$$

$$(dX_t)^2 = X_t\left([\alpha\mu + (1-\alpha)r]^2 dt^2 + \alpha^2\sigma^2 (dB_t)^2 + 2 \cdots \right)$$

$$= \alpha^2 \sigma^2 dt \cdot X_t$$

$$d\log(X) = \frac{1}{X_t} X_t [\alpha\mu + (1-\alpha) r] dt + \alpha\sigma dB_t]$$

$$+ \frac{1}{2}\frac{1}{X_t^2}[\alpha^2\sigma^2 dt] X_t^2$$

$$d\log(X) = (\alpha\mu + (1-\alpha) r) dt + \alpha\sigma dB_t =$$

$$-\frac{1}{2}\alpha^2\sigma^2 dt$$

$$d\log(X) = (\alpha\mu + (1-\alpha) r - \frac{1}{2}\alpha^2\sigma^2) dt + \alpha\sigma dB_t.$$

$$\log\frac{X_T}{X_0} = \int \cdots + \underbrace{\int \alpha \sigma dB_t}$$

in the horizont

$$V(t,x) = \sup_{\alpha} \mathbb{E}\left[U(X_T) \right]$$

$$= \sup_{\alpha} \mathbb{E}\left[g(X_T) \right]$$

$$\frac{\partial u}{\partial t} + \sup_{\alpha} \left[b \frac{\partial u}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + f \right] = 0$$

$$b = (r + \alpha(\mu - r))x$$

$$\sigma = \cancel{(\ldots)}\, x \sigma \alpha$$

$$\frac{\partial u}{\partial t} + \sup_{\alpha} \left[[r + \alpha(\mu - r)] x \frac{\partial u}{\partial x} + \frac{(\sigma\alpha)^2}{2} x^2 \frac{\partial^2 u}{\partial x^2} + f \right] = 0$$

$$\frac{\partial u}{\partial t} + rx \frac{\partial u}{\partial x} + \sup_{\alpha} \left[[\alpha(\mu - r) x \frac{\partial u}{\partial x} + \frac{\sigma^2 \alpha^2 x^2}{2} \frac{\partial^2 u}{\partial x^2} + f \right] = 0.$$

$$v(T,x) = g(x) = U(x).$$

$$U = x^p/p \quad p < 1 \quad p \neq 0.$$

candidate form

$$V(t,x) = h(t) U(x) = h(t) x^p.$$

$$\dot{h}(t) + C h(t) = 0 \qquad h(T) = 1.$$

Allocation with Consumption

$$dX_t = X_t\left(r + \alpha(\mu - r) - c\right)dt + X_t \alpha \sigma \, dB_t .$$

$$v(x) = \sup_{\alpha, c} \mathbb{E}\left[g(X_T) + \int_t^T f \, ds \right]$$

max the consumption utility

$$v(x,t) = \sup_{\alpha, c} \mathbb{E}\left[\int_t^T f \, ds \right] \qquad f = e^{-\beta t} U(c_t X_t)$$

$$\frac{\partial v}{\partial t} + \sup_{\alpha, c}\left[b \frac{\partial u}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + f \right] = 0$$

$$\frac{\partial v}{\partial t} + \sup_{\alpha, c}\left[(r + \alpha(\mu - r) - c) x \frac{\partial u}{\partial x} + \frac{x^2 \alpha^2 \sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + f \right] = 0 .$$

$$\frac{\partial v}{\partial t} + (r) x \frac{\partial v}{\partial x} + \underbrace{\sup_{\alpha, c}\left[(\alpha(\mu - r) - c) x \frac{\partial u}{\partial x} + \frac{x^2 \alpha^2 \sigma^2}{2} \frac{\partial^2 u}{\partial u^2} + f \right]} = 0 .$$

$$+ \sup_{\alpha}\left[\alpha(\mu - r) x \frac{\partial u}{\partial x} + \frac{x^2 \alpha^2 \sigma^2}{2} \frac{\partial^2 u}{\partial u^2} \right] + \sup_{c}\left[e^{-\beta t} U(c) - c x \frac{\partial v}{\partial x} \right]$$

$U(x) = x^p$

$$\frac{\partial v}{\partial t} + r x \frac{\partial v}{\partial x} +$$

DPP Classic.

$$u(t,x) = \inf_{\substack{\alpha \\ t,t+h}} \mathbb{E}\left[u(t+h, X_{t+h}) + \int_t^{t+h} f\,ds \right]$$

Taylor $\quad u(t+h, X_{t+h}) = u(t,x) + \int_t^{t+h} \frac{\partial u}{\partial t}\,dt + \int_t^{t+h} \frac{\partial u}{\partial x}\,dx + O(2)$

$$0 = \inf_{\alpha} \mathbb{E}\left[\int_t^{t+h} \frac{\partial u}{\partial t}\,dt + \int_t^{t+h} \frac{\partial u}{\partial x}\,dx + \int_t^{t+h} f\,ds \right]$$

divide by t $\qquad \frac{\partial u}{\partial t}\Big|_{x \text{ fixed}} \qquad X(\alpha, t)$

$h \to 0$

$$0 = \inf_{\alpha} \mathbb{E}\left[\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\dot{x} + f \right]$$

$$0 = \frac{\partial u}{\partial t} + \inf_{\alpha}\left[\frac{\partial u}{\partial x}\dot{x} + f \right]$$

$$\int_t^{t+h} f(s)\,ds = F(t+h) - F(t)$$

$$\frac{1}{h}\int_t^{t+h} f(s)\,ds = \frac{F(t+h) - F(t)}{h} = F'(t)$$

DPP stochastic

$$u(t,x) = \inf_{\alpha} \mathbb{E}\left[\underbrace{u(t+h, X_{t+h})}_{\text{Itô}} + \int_t^{t+h} f \, ds \right]$$

$X_t = x$

$$u(t+h, X_{t+h}) = u(t,x) + \int_t^{t+h} \frac{\partial u}{\partial t} dt + \int_t^{t+h} \frac{\partial u}{\partial x} dX_t + \frac{\sigma^2}{2} \int_t^{t+h} \frac{\partial^2 u}{\partial x^2} dt.$$

$$0 = \inf_{\alpha} \mathbb{E}\left[\int_t^{t+h} \left(\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + f \right) dt + \int_t^{t+h} \frac{\partial u}{\partial x} dX_t \right]$$

$$dX_t = b\, dt + \sigma\, dB_t$$

$$0 = \inf_{\alpha} \mathbb{E}\left[\int_t^{t+h} \left(\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + f + b \frac{\partial u}{\partial x} \right) dt. + \int_t^{t+h} \sigma\, dB_t \, \frac{\partial u}{\partial x} \right]$$

- $\int_t^{t+h} \left(\sigma \frac{\partial u}{\partial x} \right)^2 ds < \infty$

divide by h

$h \to 0$

$$\cancel{0 = \frac{\partial u}{\partial t} + \inf_{\alpha} \mathbb{E}\left[\int_t \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + f + b \frac{\partial u}{\partial x} \right.}$$

- $\frac{\partial u}{\partial t} = \left. \frac{du}{dt} \right|_{x = \text{const}}$ ind α

$$\begin{cases} \dfrac{\partial u}{\partial t} + \inf_{\alpha} \left\{ b \dfrac{\partial u}{\partial x} + \dfrac{\sigma^2}{2} \dfrac{\partial^2 u}{\partial x^2} + f \right\} = 0 \\ u(T,x) = g(x). \end{cases}$$

locally int

- $\int_t^{t+h} f\, ds = F(t+h) - F(t)$

divide by h $\quad \frac{F(t+h) - F(t)}{h}$

$h \to 0$

$= F'(t) = f(t)$

$dX_t = b(t, X_t, \alpha_t)\,dt + \sigma(t, X_t, \alpha_t)\,dB_t$

$(b, \sigma) : [0,T] \times \mathbb{R} \times A$

Linear SDE $\exists!$ $\forall x$
$\sup_t (c_1, c_2) < \infty$
c_1, c_2 continuous in $[0,T]$

1) Growth $|b(t,x,\alpha)| + |\sigma(t,x,\alpha)| \leq C(1 + |x| + |\alpha|)$

2) Lipschitz $|\sigma(t,x,\alpha) - \sigma(t,y,\alpha)| \leq L(|x-y|)$

Def. Filtration :

Def. Admissible Control : A valued, F_t measurable , $\mathbb{E}\int_0^t \alpha^2 ds < \infty$ $|\alpha| < K$

$\exists!$ continuous paths $\longrightarrow$ $\mathbb{E}\left[\sup_{s<T} |X_s|^2\right] < \infty$ Doobs L^2

Example $dX_t = \alpha_t dt + B_t$

Example $dS_t = \rho S_t dt + \sigma S_t dB_t$
$dS^0_t = \mu S^0_t dt$

$\Rightarrow$

$X_t = \alpha_t X_t + (1-\alpha_t) X_t$

$X_t = \frac{\alpha_t X_t}{S_t} S_t + \frac{(1-\alpha_t)X_t}{S^0_t} S^0_t$

$dX_t = \frac{\alpha_t X_t}{S_t} dS_t + \frac{(1-\alpha)X_t}{S^0_t} dS^0_t$

Example $-C_t dt\, X_t$

rate C_t
$C_t dt$ proportion of weath in $t, t+dt$

$|C_t| < K$
$\int_0^T c^2 ds < \infty$

when taking $d(X_t)$ we substract $C_t dt X_t$.

Example. $dX_t = P_t\, q_t\, dt - c(q_t)dt$

$q_t dt$ = quantity produced

price

rate of production

weath
$dX_t = P_t q_t dt - c(q_t) dt$
price $dP_t = bP_t dt + \sigma P_t dB_t$
Quantity $dQ_t = q_t dt$

$J(\alpha) = \mathbb{E}[g(X_T)]$ well defined. is finite expectation.

g utility function well defined.

$J = \sup J(\alpha)$

$|g(x)| \leq C(1 + |x|^2)$

$\mathbb{E}|g(X_T)| \leq \mathbb{E}|X_t|^2 \leq \mathbb{E}\sup|X_t|^2 < \infty$

$J(\alpha) = \mathbb{E}\left[\int_0^T f\,dt\right]$

$\int_0^T f\,dt = cT + \int_0^T |X_t|^2 dt + \int_0^T |\alpha|^2 dt$

$|f| \leq C(1 + |x|^2 + |\alpha|^2)$

$\mathbb{E}\int_0^T \alpha^2 ds < \infty$

$\mathbb{E}[\sup|X_t|^2] < \infty$

$\mathbb{E}[|f|] \leq c + \mathbb{E}[|X|^2] + \mathbb{E}|\alpha|^2]$

$\frac{1}{T}\int_0^T |X_t|^2 dt \leq \frac{1}{T}\int_0^T \sup_t |X|^2 dt \leq \frac{\cancel{T}}{\cancel{T}} \sup|X_t|^2$

Dynamical Programming Principle

$$u(0,x) = \inf_{\alpha_t} \mathbb{E}_0 \left[g(X_T) + \int_0^T f\,dt \right] \qquad [0,T]$$

$$u(t,x) = \inf_{\alpha_s} \mathbb{E}_t \left[g(X_T) + \int_t^T f\,dt \right] \qquad [t,T]$$

$$u(t,x) = \inf_{\alpha} \mathbb{E}_t \left[g(X_T) + \int_t^{t+h} f\,ds + \int_{t+h}^T f\,ds \right]$$

$$= \mathbb{E}_t \left[\underbrace{\mathbb{E}_{t+h}\left[g(X^*_t) + \int_{t+h}^T f^*\,ds \right]}_{u(t+h,x)} + \int_t^{t+h} f^*\,ds \right]$$

$$= \mathbb{E}\left[u(t+h, X_{t+h}) + \int_t^{t+h} f^*\,ds \right]$$

$$= \inf_{\alpha_{t,t+h}} \mathbb{E}\left[u(t+h, X_{t+h}) + \int_t^{t+h} f\,ds \right]$$

HJB value function smooth.

$$\frac{\partial u}{\partial t} + \inf_{\alpha \in A} \left[\frac{\partial u}{\partial x} b + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + f \right] = 0$$

$$u(T,x) = g(x).$$

$$|b(t,0,\alpha^*)| \le c$$

$$|b(t,x,\alpha^*)| \le |b(t,x,\alpha^*) \pm b(t,0,\alpha^*)|$$

$$\le \underbrace{|b(t,0,\alpha^*)|}_{\le C} + \underbrace{|b(t,x,\alpha^*) - b(t,0,\alpha^*)|}_{L|x-0|}$$

$$\le C(1+|x|)$$

Example.

$$g(x) = x^p$$

$$b : p\alpha x + (1-\alpha)\mu x$$

$$\sigma = \sigma\alpha x.$$

$$v = \varphi(t)\, x^p$$

$$v(T,x) = x^p = g(x)$$

$$dX_t = bX_t\,dt + \sigma X_t\,dB_t \qquad x=0$$

$$\frac{\partial u}{\partial t} + \sup_{\alpha\in A}\Big[(p\alpha x + (1-\alpha)\mu x)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2\alpha^2x^2\frac{\partial^2 u}{\partial x^2} + \cancel{f^0}\Big]$$

$$\varphi(t) = e^{c(T-t)} \qquad v = e^{c(T-t)}x^p$$

$$\frac{1}{T}\int_0^T |X_s|^2\,ds \le \frac{1}{T}\int_0^T \sup_s |X_s|^2\,ds = \sup |X_s|^2$$

$$\mathbb{E}\,\frac{1}{T}\int_0^T |X_s|^2\,ds \le \mathbb{E}\Big[\sup_s |X_s|^2\Big] < \infty$$

Example $\quad dX_t = b\,dt + \sigma\,dB_t$

$\qquad\qquad dX_t = \alpha_t\,dt + dB_t$

$$J(\alpha) = \mathbb{E}\Big[g(X_T) + \int_0^T f_s\,ds\Big]$$

$$g=0 \qquad f_s = f_s + \frac{1}{2}\alpha_s^2$$

HJB $$\frac{\partial u}{\partial t} + \inf_\alpha\Big[b\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} + \Big[f(x) + \frac{1}{2}\alpha^2\Big]\Big] = 0$$

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + f(x) + \underbrace{\inf_\alpha\Big[\alpha\frac{\partial u}{\partial x} + \frac{1}{2}\alpha^2\Big]} = 0$$

$$\frac{d}{d\alpha}\Big[\alpha\frac{du}{dx} + \frac{1}{2}\alpha^2\Big] = \frac{\partial u}{\partial x} + \alpha = 0$$

$$\alpha = -\frac{\partial u}{\partial x} \qquad \frac{d^2}{d\alpha^2}\Big[\quad\Big] = 1 > 0 \ \cup$$

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + f(x) - \frac{1}{2}\Big(\frac{\partial u}{\partial x}\Big)^2 = 0$$

Hopf $\quad z = \exp \lambda u$

$\qquad u = \frac{1}{\lambda}\ln z.$

Feyman Kac

$$\frac{\partial u}{\partial t} + \mu \frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + Vu + f = 0$$

$$u(x,t) = \mathbb{E}_t\left[\int_t^T e^{-\int_t^s V\,dz} f\,ds + e^{-\int_t^T V\,dz} g(X_T)\right]$$

for $V=0$

$$\frac{\partial u}{\partial t} + \mu \frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + f = 0$$

$dX_t = \mu\,dt + \sigma\,dB_t$

$u(X_t, t)$, $f(X_t, t)$

Itô w

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dX_t + \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}(dX_t)^2\right]$$

$$du = \left(\frac{\partial u}{\partial t} + \mu\frac{\partial u}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}\right)dt + \frac{\partial u}{\partial x}\sigma\,dB_t$$

$$u(X_T, T) = u(X_t, t) + \int_t^T -f\,ds + \int_t^T \sigma\frac{\partial u}{\partial x}dB_t$$

$$g(X_T) = u(X_t, t) - \mathbb{E}_t\left[\int_t^T f\,ds\right] + 0$$

$$u(X_t, t) = g(X_T) + \mathbb{E}_t\left[\int_t^T f\,ds\right]$$

$$= \mathbb{E}_t\left[g(X_T) + \int_t^T f\,ds\right]$$

$$\frac{\partial u}{\partial t} + \mu \frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} - Vu + f = 0$$

$$dX_t = \mu\, dt + \sigma\, dB_t$$

$$du = \underbrace{\left(\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial u}{\partial x} \right)}_{Vu - f} dt + \frac{\partial u}{\partial x} \sigma\, dB_t$$

$$du - Vu\, dt = -f\, dt + \sigma \frac{\partial u}{\partial x} dB_t$$

$$I\, du - I V u\, dt = I_s(\qquad)$$

$$d(Iu) = I(\qquad)$$

$$I = e^{-\int V dz} \qquad \frac{I_0}{I_T} = e^{\int^T V dz}$$

$$u(X_T, T) = u(x_0, 0) \frac{I_0}{\cancel{I_T}} - \cancel{\frac{1}{I_T}} \int_0^T I_s f\, dt + \cancel{\frac{1}{I_T}} \int_0^T \sigma \partial u\, dB_s I_s$$

$$\mathbb{E}\left[\underbrace{g(X_T) \frac{I_T}{I_0}}_{} + \frac{1}{I_0} \int_0^t f\, dt \right] = u(x_0, 0)$$

$$e^{-\int_0^T V dz} g(X_T) + \int_0^T \frac{I_s}{I_0} f\, ds$$

$$\downarrow$$

$$e^{-\int_0^s V dz} f\, ds$$

Conditions

$\mathbb{E} X_0^2 < \infty$

Growth $\quad |b(t,x,\alpha)| \le 1 + |x| + |\alpha|$

Lipschitz $\quad |b(t,x,\alpha) - b(t,y,\alpha)| \le L|x-y|$

$$|b(t,x,\alpha)| \le |b(t,0,\alpha)| + |b(t,x,\alpha) - b(t,0,\alpha)|$$
$$\le C + L|x-0|$$
$$\le \sup C, L \ (1 + |x|)$$

or

$$b = x(\rho-\mu)\alpha + \mu x$$

$$|b| \le |x| \, |\rho-\mu| \, |\alpha| + |\mu| \, |x|$$
$$\le |x| \, (c^+)$$
$$\le c^+ + C^+|x| = C^+(1+|x|) \le C^+(1+|x|+|\alpha|)$$

$$|f| \le 1 + |x|^2 + |\alpha|^2$$

$$|g| \le 1 + |x|^2$$

Example

$$V(t,x) = \sup_{\alpha} \mathbb{E}\left[g(X_T) + \int_t^T f(X_s)\, ds \right]$$

$f := 0$

$g(x) = x^p$

$$dX_s = \left[(\rho-\mu)\alpha + \mu\right] X_s\, ds + X_s \sigma \alpha\, dB_s$$

$$X_T = x \exp\left[\int_t^T (\rho-\mu)\alpha + \mu - \frac{\sigma^2\alpha^2}{2}\, ds + \int_t^T \sigma\alpha\, dB_s \right]$$

a) $$V(t,x) = \sup_{\alpha} \mathbb{E}\left[x^p e^{p\int_t^T \, ds + p\int_t^T \sigma\alpha\, dB_s} \right]$$

$$= x^p \sup_{\alpha} \underbrace{\left[e^{p\int_t^T \dots\, ds} \right]}_{\#} \underbrace{\mathbb{E}\left[e^{p\int_t^T \sigma\alpha\, dB_s} \right]}_{\text{Momentum Generator}}$$

Momentum Generator $N\left(0, \int_t^T p^2\sigma^2\alpha^2\, ds\right)$

$$= x^p \sup_{\alpha} e^{p\int_t^T \dots\, ds}\, e^{\frac{p^2}{2}\int_t^T \alpha^2\sigma^2\, ds} \qquad e^{\lambda m + \frac{\lambda^2}{2}\sigma^2}$$

$$= x^p \underbrace{\sup_{\alpha} e^{\int_t^T p[\rho-\mu]\alpha + p\mu + p(p-1)\frac{\sigma^2\alpha^2}{2}\, ds}}_{C_{t,T}}$$

$$V(t,x) = C_{t,T}\, x^p \qquad \sup_{\alpha} e^{f} = e^{\sup_{\alpha} f}$$

b) $$C_{t,T} = C_{t,t+h} \cdot C_{t+h,T} \qquad \sup_{\substack{\alpha \\ [t,T]}} = \sup_{\substack{\alpha \\ [t,t+h]}} + \sup_{\substack{\alpha \\ [t+h,T]}}$$

$$V(t,x) = \sup_{\alpha} \mathbb{E}\left[V(t+h, X_{t+h}) \right] =$$

$$= x^p C_{t,T} = \sup_{\alpha} \mathbb{E}\left[C_{t+h,T} (X_{t+h})^p \right] = x^p C_{t+h,T} \sup_{\alpha} \mathbb{E}\underbrace{\left[e^{\int_t^{t+h}} \right]}_{C_{t,t+h}}$$

$$C_{t,T} = \sup_{\alpha} e^{\int_t^T ds} = e^{\sup_{\alpha}\int_t^T ds}$$

$$\frac{dC_{t,T}}{dt} = \frac{d}{dt}\left(\sup \int_t^T ds\right) C_{t,T}$$

$$\frac{d}{dt}\left(F^*(T) - F^*(t)\right) C_{t,T}$$

$$- f^*(t)\, C_{t,T} .$$