Definition of Brownian Motion

Is a Process B_t

- $B_0 = 0$
- B_t is continuous $\mathbb{P}$.a.s
- Independent increments ($W_t - W_s \perp\!\!\!\perp F_s$)
  and Stationary increments ($W_t - W_s \sim N(0, t-s)$ $\forall t,s$)

SDE

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t$$

$$dX_t^i = b^i(t, X_t)\,dt + \sigma^{ij}\,dB_t^j \qquad \begin{array}{l} 1 \le i \le n \\ 1 \le j \le d \end{array}$$

Definition Strong Solution

Is a stochastic process X s.t

- $X_t \subset F_t$ adapted to the BM
- Riemann and Itô int. well defined

$$\int_0^T E\, b_s^2\, ds < \infty$$

- X_t is a function of the sample path of B_t, b, σ.

" Different realization of B_t => Different Solution"

Definition Weak solution

- Find the distribution of X_t (moments)
- Path of B_t NOT necessary.

Any Solution is called Diffussion

<u>Theorem</u> $\exists!$ Solution (Strong) on $[0,T]$

- $E[X_0^2] < \infty$, X_0 ind of B_t.

- Growth cond. $|\sigma(t,x)| + |b(t,x)| \leq K(1+|x|)$

  - continuous σ, b in $[0,T]$ ✓

- Lipschitz cond on space.

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y|$$

<u>Itô Lemma</u>

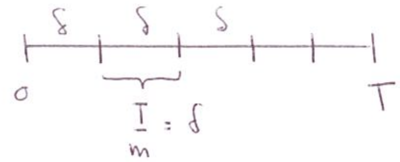
Compact
$$du(t,X_t) = \frac{\partial}{\partial t} u(t,X_t)\, dt + \nabla u \cdot dX_t + \frac{1}{2} \mathrm{Tr}\big[\sigma \sigma^T \underbrace{u_{xx}}_{\text{matrix spatial derivatives } 2^{nd} \text{ order}}\big]$$

<u>Why Simulate:</u>

Only few cases have exact solution.

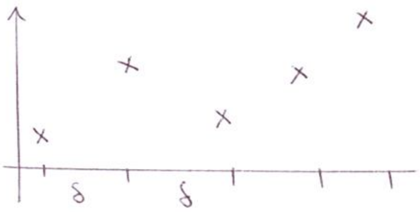
Discretization Schemes

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t$$



We want to build a Discrete Time Markov chain $\bar{X}^\delta = (\bar{X}^\delta_{\ell\delta})$

$\ell = \{0, 1, 2, \dots n$



$\bar{X}^\delta_{\ell\delta}$

$\bar{X}^\delta_1 \quad \bar{X}^\delta_2$

i) $\bar{X}^\delta$ easy to simulate

ii) Law $\bar{X}^\delta_T \simeq$ Law X_T

(at end)

$$E[f(\bar{X}^\delta_T)] \simeq E[f(X_T)]$$

Euler Scheme

Replace differentials by differences

Only need simulate W_δ, $W_{2\delta} - W_\delta$ increments of BM

$$\bar{X}^\delta_{(\ell+1)\delta} = \bar{X}^\delta_{\ell\delta} + b(\ell\delta, \underbrace{\bar{X}^\delta_{\ell\delta}}_{\text{previous simulation}})\,\delta$$

$$+ \sigma(\ell\delta, \bar{X}^\delta_{\ell\delta}) \underbrace{\left[B_{\ell\delta} - B_{(\ell-1)\delta} \right]}_{\sim N(0,\delta)}$$

$\sqrt{\delta}\, N(0,1)$

No need of derivatives of b or σ.

Now the drift parts goes as δ
but the random part as $\sqrt{\delta}$

In distributional sense

$$E\left[\left(\int_{l\delta}^{(l+1)\delta} b\, ds\right)^2\right] \sim b^2(\delta)^2$$

b is fixed in this interval as $b(l\delta, \bar{X}^{\delta}_{l\delta})$

and

$$E\left[\left(\int_{l\delta}^{(l+1)\delta} \sigma\, dB\right)^2\right] \sim \sigma^2\delta$$

$$= E\left[\int_{l\delta}^{(l+1)\delta} \sigma^2 ds\right]$$

σ fixed as $\sigma(l\delta, \bar{X}^{\delta}_{l\delta})$

— Euler converges

strongly as $\delta^{1/2}$

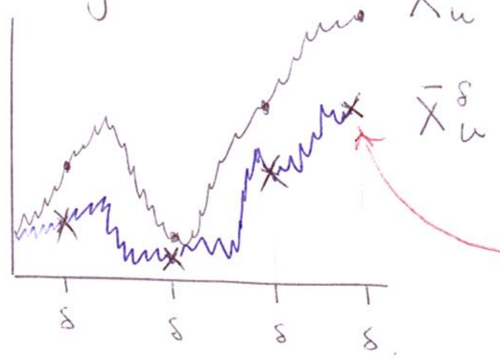
weakly as δ

— Milstein converges

strongly δ (better) but need derivative σ

but Milstein is very bad for $\dim > 1$.

Convergence in L^p



$$X_u - \bar{X}^\delta_u \qquad u \in [0, T]$$

continuous

This points are Euler or Milstein

We want that the evolution of $\bar{X}^\delta$ as

$$d\bar{X}^\delta_t = b(l\delta, \bar{X}^\delta_{l\delta})\,dt + \sigma(l\delta, \bar{X}^\delta_{l\delta})\,dB_t$$

in $[l\delta, (l+1)\delta]$

$z(s) = l\delta$ when $s \in [l\delta, (l+1)\delta]$ so, is the left time point

$$\bar{X}^\delta_t = \int_0^t b(z(s), \bar{X}^\delta_{z(s)})\,ds + \int_0^t \sigma(z(s), \bar{X}^\delta_{z(s)})\,dB_s$$

$$\Rightarrow X_u - \bar{X}^\delta_u = \int_0^u b(s, X_s) - b(z(s), \bar{X}^\delta_{z(s)})\,ds + \int_0^u \sigma(s, X_s) - \sigma(z(s), \bar{X}^\delta_{z(s)})\,dB_s$$

$$|a+b|^{2p} \leq 2^{2p-1}\left(|a|^{2p}+|b|^{2p}\right)$$

we will show that

$$E \sup_{u\in[0,T]} |X_t - X^n_t|^{2p} \leq \frac{K(T)}{n^p}$$

$\to$ Lipschitz b, σ, p

$$E\left[\sup_u |X_u - \bar{X}^\delta_u|^{2p}\right] \leq 2^{2p-1} E\left[\sup_u \left|\int_0^u b - b(z,\bar{X}^\delta_z)\,ds\right|^{2p} + \left|\int_0^u \sigma(s,X_s) - \sigma(z,\bar{X}^\delta_z)\,d\right.\right.$$

sup of sum of positive functions

$$= \underbrace{2^{2p-1} E \sup_u \left|\int_0^u b(s,X_s) - b(z(s),\bar{X}^\delta_{z(s)})\,ds\right|^{2p}}_{\mathrm{I}}$$

$$+ \underbrace{2^{2p-1} E \sup_u \left|\int_0^u \sigma(s,X_s) - \sigma(z(s),\bar{X}^\delta_{z(s)})\,dB_s\right|^{2p}}_{\mathrm{II}}.$$

I) bring abs value inside get a positive process so, bounded by $u=T$

$$\leq 2^{2p-1} E \sup_u \left[\int_0^u |b(s,X_s) - b(z,\bar{X}^\delta_z)|\,ds\right]^{2p}$$

$$\cancel{= 2^{2p-1} E \left[\int_0^T |b(s,X_s) - b(z,\bar{X}^\delta_z)|\,ds\right]}$$

First Apply Hölder

$\frac{1}{2p} + \frac{1}{q} = 1$

$\frac{1}{q} = 1 - \frac{1}{2p} = \frac{2p-1}{2p}$

$$\leq 2^{2p-1} E \sup_u \left[\left(\int_0^u |b-b|^{2p}\,ds\right)^{1/2p} \underbrace{\left(\int_0^u 1^q\,ds\right)^{1/q}}_{u^{1/q}}\right]^{2p}$$

$$\leq 2^{2p-1} \, \underline{E} \sup_u \left[\left(\int_0^u |b - b|^{2p} \right)^{\cancel{2p/2p}} u^{2p/q} \right] \qquad q = \frac{2p-1}{2p}$$

positive

$$\leq 2^{2p-1} \, T^{2p-1} \, \underline{E} \int_0^T |b(s, X_s) - b(z(s), \bar{X}^\delta_{z(s)})|^{2p} ds$$

Now $|b(s, X_s) - b(z(s), \bar{X}^\delta_{z(s)})|^{2p}$

add and subst $\pm b(s, X_{z(s)})$

$\pm b(z(s), X_{z(s)})$

$$|b(s, X_s) - b(s, X_{z(s)}) + b(s, X_{z(s)}) - b(z(s), X_{z(s)}) + b(z(s), X_{z(s)}) - b(z(s), \bar{X}^\delta_{z(s)})|^{2p}$$

$$\leq 2^{2p-1} |b(s, X_s) - b(s, X_{z(s)}) + b(s, X_{z(s)}) - b(z(s), X_{z(s)})|^{2p}$$

$$+ 2^{2p-1} |b(z(s), X_{z(s)}) - b(z(s), \bar{X}^\delta_{z(s)})|^{2p}$$

$$\leq (2^{2p-1})^2 \left(|b(s, X_s) - b(s, X_{z(s)})|^{2p} + |b(s, X_{z(s)}) - b(z(s), X_{z(s)})| \right.$$

$$+ 2^{2p-1} |b(z(s), X_{z(s)}) - b(z(s), \bar{X}^\delta_{z(s)})|^{2p}$$

for terms with ~~same~~ different time $\longrightarrow$ Growth cond $\leq K(1+|x|)(t-s)^{\gamma}$

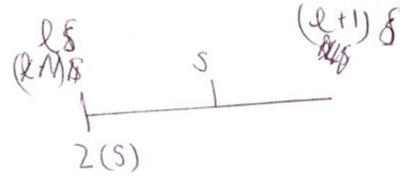
" " " " x $\longrightarrow$ Lipschitz. $\leq L|x-y|$

$$|b(s,X_s) - b(s,X_{z(s)})|^{2p} \leq L^{2p}|X_s - X_{z(s)}|^{2p}$$

$$|b(s,X_{z(s)}) - b(z,X_z)|^{2p} \leq K^{2p}(1+|X_{z(s)}|)^{2p}(z-s)^{\gamma 2p}$$

$$|b(z,X_z) - b(z,\bar{X}^{\delta}_z)|^{2p} \leq L^{2p}|X_{z(s)} - \bar{X}^{\delta}_{z(s)}|^{2p}$$

now $z(s) - s \leq \delta$



$$|X_{z(s)} - \bar{X}^{\delta}_{z(s)}|^{2p} \leq \sup_r |X_r - \bar{X}^{\delta}_r|^{2p}$$

To convert from dB_s to ds
we need

<u>Burkholder Davis lemma</u>

$$k_m \, E\left[\int_0^T \phi^2 \, ds\right]^m \leq E\left[\sup_{0\leq s\leq T} \left|\int_0^s \phi_s \, dB_s\right|^{2m}\right] \leq K_m \, E\left[\int_0^T \phi^2 ds\right]^m$$

Doob's L^p Inequality

$$E \sup_{0\leq s\leq T} |M_s|^p \leq \left(\frac{p}{p-1}\right)^p E|M_T|^p \qquad M_s \text{ Martingale}$$

$$\Rightarrow \quad \Big|\underbrace{\int_0^u \sigma(s, X_s) - \sigma(z, \bar{X}_z^\delta) \, dB_s}_{\text{Martingale (Itô)}}\Big|^{2p}$$

$$\Rightarrow \quad E \sup_s \left|\int_0^u \sigma(s, X_s) - \sigma(z, \bar{X}_z^\delta) \, dB_s\right|^{2p} \leq K_m \, E\left[\int_0^T |\sigma - \sigma|^2 ds\right]^p$$

Hölder

$|\langle u,v\rangle|$

$\leq \|u\|_p \|v\|_q$

p [illegible] q

$v = 1$

$u = |\sigma - \sigma|^2$

$$\leq K_m \, E\left[\underbrace{\left(\int_0^T |\sigma - \sigma|^{2p} ds\right)^{1/p}}\underbrace{\left(\int_0^T 1^q \, ds\right)^{1/q}}_{T^{p/q}}\right]^p$$

$$= T^{p/q} K_m \, E\left[\int_0^T |\sigma - \sigma|^{2p} ds\right]$$

continue as with b.

get same.

we get

$$E \sup_u |X_u - \bar{X}_u^\delta|^{2p} \leq C_T^{(p)} E \int_0^T |X_s - X_{z(s)}|^{2p} ds$$

$$+ C_T^{(p)} E \int_0^T (1 + |X_{z(s)}|)^{2p} \delta^{2p\alpha}$$

$$+ C_T^{(p)} E \sup_r |X_{z(s)} - \bar{X}^\delta_{z(s)}|^{2p}$$

integrate $\int_0^T$

use lemma

Thm 7.9

$$E|X_t|^{2m} \leq 1 + E|X_0|^{2m} e^{Ct}$$

- $E|X_t - X_s|^{2m} \leq C(1 + E|X_0|^{2m})(t-s)^m$

$$E \sup_{0 \leq t \leq T} |X_t|^{2m} \leq C(1 + E|X_0|^{2m}) e^{CT}$$

- $E \sup_{0 \leq t \leq T} |X_t^n|^{2m} \leq C(1 + E|X_0|^{2m}) e^{CT}$

$$F(T) \leq C_T^{(p)} \left(\delta^p + \delta^{2\alpha p} + \int_0^t F(s) ds \right)$$

Gronwall

$$\text{if } g(t) \leq \alpha(t) + \beta \int_0^t g(s) ds$$

$$\Rightarrow g(t) \leq \alpha(t) + \beta \int_0^t \alpha(s) e^{\beta(t-s)} ds$$

$$F(T) \leq C_T^{(p)} (\delta^p + \delta^{2\alpha p}) + C_T^{(p)} \int_0^t (\delta^p + \delta^{2\alpha p}) e^{(t-s)} ds$$

$$\Rightarrow \quad F(T) \leq C_T^{(p)} \left(\delta^p + \delta^{2\alpha p} \right)$$

Bigger exponent $\Rightarrow$ slower rate

$$= C_T^{(p)} \quad \delta^{2 \min\left(\frac{1}{2}, \alpha\right) p}.$$

so we care about the smaller (faster)

Convergence Almost surely Euler.

$$\delta = \frac{T}{m} \qquad \frac{1}{\delta^\theta} \sup |X_t - \bar{X}^\delta_t| \longrightarrow 0$$

$$Z^m = m^\theta \sup_{0 \le t \le T} |X_t - \bar{X}^\delta_t| \qquad \delta \longrightarrow 0 \text{ means } m \longrightarrow \infty$$

Chebyshev

$$\mathbb{P}(Z^m \geq \epsilon) \leq \frac{\cancel{\mathrm{Var}}(Z^m)^{2p}}{\epsilon^{2p}} =$$

$$\boxed{\begin{array}{l} \mathbb{P}(|X-\mu| \geq \epsilon) = \dfrac{\mathrm{var}\, X}{\epsilon^2} \\ \mu = 0 \\ \mathbb{P}(|X| \geq \epsilon) \leq \dfrac{E X^2}{\epsilon^2} \\ \text{Markov} \\ \mathbb{P}(X \geq \epsilon) \leq \dfrac{E X}{\epsilon} \\ \mathbb{P}(X^{2p} \geq \epsilon^{2p}) \leq \dfrac{E X^{2p}}{\epsilon^{2p}} \\ \Rightarrow \mathbb{P}(|X| \geq \epsilon) \leq \dfrac{E X^{2p}}{\epsilon^{2p}} \end{array}}$$

$$\sum_{m \geq 1} \mathbb{P}(Z^m \geq \epsilon) \leq \sum_{m \geq 1} \frac{\cancel{\mathrm{Var}}(Z^m)^{2p}}{\epsilon^{2p}}$$

$$= \sum_{m \geq 1} \frac{1}{\epsilon^{2p}} \cancel{\mathrm{Var}} \left[m^\theta \sup |X_t - \bar{X}^\delta_t| \right]^{2p}$$

$$= \sum_{m \geq 1} \frac{m^{2p\theta}}{\epsilon^{2p}} \underbrace{E \left[\sup |X_t - \bar{X}^\delta_t| \right]^{2p}}_{L^p \text{ convergence}}$$

$$\leq \sum_{m \geq 1} \frac{m^{2p\theta}}{\epsilon^{2p}} C^{(p)}_T \Big(\underbrace{\frac{T}{m}}_{\delta}\Big)^{2\beta p} \leq \frac{C^{(p)}_T}{\epsilon^{2p}} \sum_{m \geq 1} m^{2\theta p - 2\beta p} < \infty$$

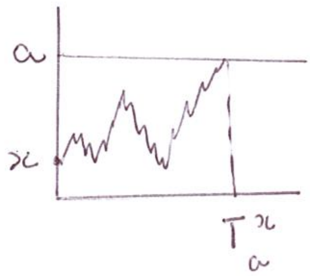
since $2p\theta - 2\beta p < -1$

Borel Cantelli

$Z^m \longrightarrow 0$. a.s

First Hitting Time

$(B_t^x)_{t \geq 0}$ BM starting in x



$$T_a^x = \inf\{t > 0,\ B_t^x \geq a\}$$

$$M_t = \sup_{0 \leq s \leq t} B_t$$

$$\mathbb{P}(T_a^x \leq t) = \mathbb{P}\left(\sup_{0 \leq s \leq t} B_s^x \geq a\right)$$

Strong Markov Property

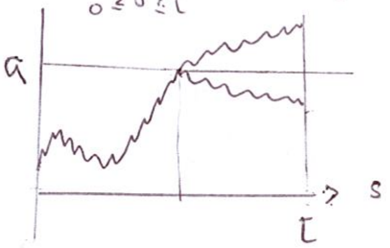
If Z is a stopping time a.s finite ($\mathbb{P}(Z < \infty) = 1$)

Then $B_{Z+t}^x - B_Z^x$ is a SBM starting in 0 and $\perp\!\!\!\perp F_Z$

Reflection Principle

$$\mathbb{P}(T_a^x \leq t) = \underline{\mathbb{P}\left(\sup_{0 \leq s \leq t} B_s \geq a\right) = 2\,\mathbb{P}(B_t^x \geq a)}$$

$$\mathbb{P}\left(\sup_{0 \leq s \leq t} B_s^x \geq a\right) = \mathbb{P}\left(\sup_s B_s^x \geq a,\ B_t^x \geq a\right) + \mathbb{P}\left(\sup_s B_s^x \geq a,\ B_t^x < a\right)$$



$$= \mathbb{P}(B_t^x \geq a) + \underbrace{\mathbb{P}(T_a^x \leq t,\ B_t^x < a)}$$

$$+ \mathbb{E}\left[\mathbb{1}_{T_a^x \leq t}\, \mathbb{1}_{B_t^x - a < 0}\right]$$

$$+ \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{T_a^x \leq t}\, \mathbb{1}_{B_t^x - a < 0} \,\middle|\, F_{T_a^x}\right.\right.$$

$$+ \mathbb{E}\left[\mathbb{1}_{T_a^x \leq t}\left[\mathbb{E}\, \mathbb{1}_{B_t^x - B_{T_a^x}^x < 0} \,\middle|\, F_{T_a^x}\right.\right.$$

$$+ \mathbb{E}\left[\mathbb{1}_{T_a^x < t}\ \underbrace{\mathbb{E}\, \mathbb{1}_{B_t^0 < 0}}_{1/2}\right]$$

$$+ \frac{1}{2}\,\mathbb{P}(T_a^x \leq t)$$

$B_t^x - B_{T_a^x}^x$ SBM start in 0

ind of $F_{T_a^x}$

$t > T_a^x$

$t > s$

Corollary $\quad \mathbb{P}(T_a^x \in dt) = \frac{a-x}{\sqrt{2\pi t^3}} \exp \frac{(a-x)^2}{2t}$

By Reflection Principle

$$\mathbb{P}(T_a^x \le t) = 2\,\mathbb{P}(B_t^x \geq a)$$

$$= 2\int_a^\infty \frac{1}{\sqrt{2\pi t}}\, e^{\frac{-(z-x)^2}{2t}}\, dz$$

$$= -2\int_{+\infty}^{a} \frac{1}{\sqrt{2\pi t}}\, e^{-(z-x)^2/2t}\, dz$$

$u = \frac{z-x}{\sqrt{t}}$

$du = \frac{dz}{\sqrt{t}}$

$$= -2\int_{+\infty}^{\frac{a-x}{\sqrt{t}}} \frac{\cancel{\sqrt{t}}}{\sqrt{2\pi \cancel{t}}}\, e^{-u^2/2}\, du$$

$$= \frac{-2}{\sqrt{2\pi}} \int_{+\infty}^{\frac{a-x}{\sqrt{t}}} e^{-u^2/2}\, du$$

$$\frac{d}{dt}\int_x^y e^{-u^2/2}\, du = \frac{d}{dt}\left[F(y) - F(x) \right]$$

F is antiderivative.

$$= \frac{dF(y)}{dy}\frac{dy}{dt} - \frac{dF(x)}{dx}\frac{dx}{dt}$$

$$= e^{-y^2/2}\frac{dy}{dt} - e^{-x^2/2}\frac{dx}{dt}$$

$$\Rightarrow \quad d\,\mathbb{P}(T_a^x \le t) = \frac{-2}{\sqrt{2\pi}} \frac{d}{dt}\left[F\left(\frac{a-x}{\sqrt{t}}\right) - F(\infty) \right]$$

$$= \frac{-2}{\sqrt{2\pi}}\, e^{\frac{1}{2}\left(\frac{a-x}{\sqrt{t}}\right)^2} \frac{d}{dt}\left(\frac{a-x}{\sqrt{t}}\right) \underbrace{- \lim_{r\to\infty} e^{-r^2/2} \frac{dr}{dt}}_{\text{exponential goes} \to 0 \text{ faster}}$$

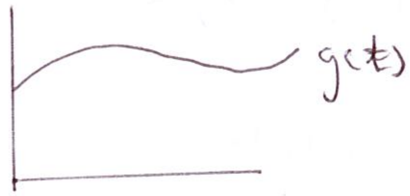
$$= \frac{-\cancel{2}\left(\frac{-1}{\cancel{2}}\right)(a-x)}{\sqrt{2\pi}\; t^{3/2}}\, e^{(a-x)^2/2t} \quad \checkmark$$

Simulation of Hitting Times

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t$$

$\bar{X}^{\delta}_{l\delta}$ is Euler Approx

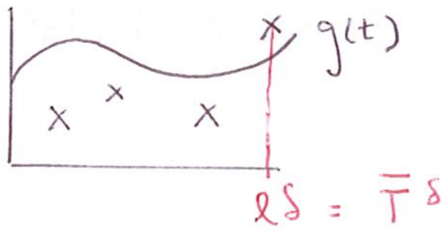
Approx $T = \inf\{ t \geq 0 ;\ X_t \geq g(t) \}$ $\quad$ $g(t)$ is cont smooth



1) First Method

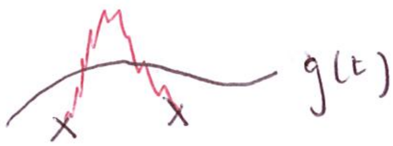
$$\bar{T}^{\delta} = \inf\{ l\delta \geq 0 \ ;\ \bar{X}^{\delta}_{l\delta} \geq g(t) \}$$

smallest discrete time when Euler is above $g(t)$



Advantage: No extra cost to simulate
Disadvantage: Easy to over estimate.

THIS COULD HAPPENED!



2) 2nd Method: Boundary Correction

$$\bar{T}^{\delta} = \inf \{ l\delta > 0 ; \; \bar{X}^{\delta}_{l\delta} \geq g(l\delta) \}$$

$$\text{or} \quad d(X^{\delta}_{l\delta}, g(l\delta)) \leq C_0 \sqrt{\delta}\, \sigma(l\delta, \bar{X}^{\delta}_{l\delta})$$

1rst Time when
when Euler cross explicity
or when the distance (Euler, $g(l\delta)$) is $\propto$ strenght of the noise

Advantage: No extra simulations
corrected over estimations

3) 3rd Method Brownian Bridge.

Monte Carlo for PDE

$$dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} (dX_t)^2$$

$$dX_t = b\, dt + \sigma\, dB_t \qquad b(t, X_t)$$

$$(dX_t)^2 = \sigma^2 dt \qquad \sigma(t, X_t)$$

$$F(T, X_T) = \Psi(X_T)$$

$$= \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial x} \left[b\, dt + \sigma\, dB_t \right] + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \sigma^2 dt$$

$$= \underbrace{\left[\frac{\partial F}{\partial t} + b \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 F}{\partial x^2} \right]}_{=0} dt + \sigma \frac{\partial F}{\partial x} dB_t$$

$$dF = \sigma \frac{\partial F}{\partial x} dB_t$$

$$\underbrace{F(T, X_T)}_{\Psi(X_T)} = F(t, X_t) + \underbrace{\int_t^T \sigma \frac{\partial F}{\partial x} dB_t}_{\text{Itô}}$$

$$\mathbb{E}\left[\Psi(X_T) \mid F_t \right] = \mathbb{E}\left[\underbrace{F(t, X_t)}_{\text{cte}} \mid F_t \right] + 0$$

$$\mathbb{E}\left[\Psi(X_T) \mid F_t \right] = F(t, X_t)$$

$\uparrow$

$X_t = x$

Feynmann Kac Theorem

Ψ grows at most polynomially at ∞

$$\frac{\partial F}{\partial t} + LF = 0$$

$$F(T, x) = \Psi(x)$$

has a $C^{1,2}$ solution $[0,T] \times \mathbb{R}^d$

$$F(t, x) = \mathbb{E}\left[\Psi(X_T) \mid X_t = x \right]$$

$dX_t = b\,dt + \sigma\, dB_t$

Example: Black Scholes (Δ Hedging)

$$\Pi(t, X_t) = \underbrace{V(t, X_t)}_{\text{option price}} - \Delta \underbrace{X_t}_{\text{asset}}$$

$$d\Pi = dV - \Delta dX_t$$

$$= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} dX_t^2 - \Delta dX_t$$

$$= \frac{\partial V}{\partial t} dt + b \frac{\partial V}{\partial x} dt + \sigma \frac{\partial V}{\partial x} dB_t + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} \sigma^2 dt - \Delta dX_t .$$

$$\cancel{= \left(\frac{\partial V}{\partial t} + b \frac{\partial V}{\partial x} + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} \sigma^2 \right) dt + \sigma \frac{\partial V}{\partial x} dB_t - \Delta \left(b\, dt\, X_t + X_t \sigma dB_t \right)}$$

$$\cancel{= \frac{\partial V}{\partial t} + b \frac{\partial V}{\partial x} + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} \sigma^2 - \Delta b}$$

$$= \left(\frac{\partial V}{\partial t} + \cancel{\frac{\partial V}{\partial x}} + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} \sigma^2 \right) dt + \left(\frac{\partial V}{\partial x} - \Delta \right) dX_t .$$

choose $\Delta = \frac{\partial V}{\partial x}$ cancell ramdomness

so Π should have the risk free rate growth

$$d\Pi = r \Pi dt$$

$$\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} \sigma^2 = r \left(V - \frac{\partial V}{\partial x} X_t \right)$$

$$\frac{\partial V}{\partial t} + r X_t \frac{\partial V}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} - rV = 0$$

Black Scholes PDE.

The change

$$\tilde{V} = e^{-r\Delta t} V = \cancel{e^{-r(T-t)} V}$$

$$\cancel{d\tilde{V} = -r e^{-r(T-t)} V dt = r\tilde{V} dt}$$

$$\frac{d\tilde{V}}{dt} = -r\tilde{V} + e^{-r\Delta t}\frac{dV}{dt}$$

$$e^{-r\Delta t}\left[\frac{\partial V}{\partial t} + r X_t \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} - rV\right] = 0$$

$$= \frac{\partial \tilde{V}}{\partial t} + r X_t \frac{\partial \tilde{V}}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 \tilde{V}}{\partial x^2} = 0$$

** No Drift now.

$$\tilde{V} = \mathbb{E}_Q\left[\tilde{\Psi}(X_T) \mid X_T = x\right]$$

Risk free measure.

Brownian Motion with Drift. Hitting Time distribution

$\tilde{T}_a$ is $\inf\{ t \geq 0 ; B_t - qt \geq a \}$

$$\mathbb{P}(\tilde{T}_a \leq t) = \mathbb{E}_P\left[\mathbb{1}_{\tilde{T}_a \leq t}\right] = \mathbb{E}_Q\left[\mathbb{1}_{T_a \leq t} M_{T_a}\right] = \mathbb{P}_Q(T_a \leq t)$$

prob $\tilde{B}_t$

prob B_t (no drift) to reach a (but in Q)

$$M_{T_a} = e^{q\int_0^{T_a} dB_s - \frac{q^2}{2}\int_0^{T_a} ds}$$

$$= e^{qB_{T_a} - \frac{q^2}{2} T_a} \qquad = \int_0^t \mathrm{pdf}(T_a)\, e^{qa - \frac{q^2}{2}s}\, ds$$

$$= \int_0^t \frac{a}{\sqrt{2\pi s^3}} e^{-a^2/2s} e^{qa - \frac{q^2 s}{2}} ds \quad ; \quad -\frac{q^2 s}{2} + qa - \frac{a^2}{2s} = -\frac{1}{2s}\left[a^2 - 2qas + q^2 s^2\right]$$

$$= -\frac{1}{2s}\left[a - qs\right]^2$$

$$= \int_0^t \frac{a}{\sqrt{2\pi s^3}} e^{-\frac{1}{2s}[a-qs]^2} ds$$

$\Rightarrow$ pdf of $\tilde{T}_a$ $\quad \frac{a}{\sqrt{2\pi s^3}} e^{-\frac{1}{2s}[a-qs]^2}$

Hitting Time of $\hat{B}_t$ reach a

BM with drift $\hat{B}_t = B_t - qt$

$$\frac{a}{s^{3/2}} = \frac{d}{ds}\left(\frac{-2a}{\sqrt{s}}\right) = \frac{d}{ds}\left(\frac{qs-a}{\sqrt{s}} - \frac{(qs+a)}{\sqrt{s}}\right) = \frac{d}{ds}\left(\frac{qs-a}{\sqrt{s}}\right) - \frac{d}{ds}\left(\frac{qs+a}{\sqrt{s}}\right)$$

$$= \int_0^t \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2s}[a-qs]^2} \left[d\left(\frac{qs-a}{\sqrt{s}}\right) - d\left(\frac{qs+a}{\sqrt{s}}\right)\right] \quad ;$$

$$[a-qs]^2 = a^2 + q^2 s^2 - 2qs \pm 4qs = [a+qs]^2 - 4qs$$

$$= \Phi\left(\frac{a-qt}{\sqrt{t}}\right) + \int_0^t \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2s}(a+qs)^2 - \frac{1}{2s}4qs} d\left(\frac{qs+a}{\sqrt{s}}\right)$$

$$= \cdots + e^{2q}\, \Phi\left(\frac{qt+a}{t}\right)$$

$$\mathbb{P}(T_a^x \leq t) = \mathbb{P}(\sup_u B_u^x \geq a)$$

reflection

$$= 2\,\mathbb{P}(B_t^x > a) = \mathbb{P}(|B_t^x| > a)$$

$$\Rightarrow \mathbb{P}(T_a^x \leq t) = \mathbb{P}(|B_t^x| > a)$$

folded normal

yee that ~~$\mathbb{P}(|B_t|...)$~~

$$|B_t| \overset{d}{\sim} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-\mu)^2}{2t}} + \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x+\mu)^2}{2t}}.$$

$$\mathbb{P}(|B_t| > a) = 1 - \mathbb{P}(|B_t| < a)$$

$$= 1 - \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x+\mu)^2}{2t}} + \frac{e^{-\frac{(x+\mu)^2}{2t}}}{\sqrt{2\pi t}} \, dt$$

$$\frac{d\mathbb{P}}{da}\underbrace{(|B_t| > a)} = \frac{d}{da}\mathbb{P}(T_a < t)$$

$$\frac{d}{da}(1 - \mathbb{P}(|B_t| < a)) = -\frac{d\mathbb{P}}{da}(|B_t| < a) =$$