Fundamentals of Time Series

Victor A. Bernal Arzola

Based on:

Prof. Zivot open ClassNotes

September, 2015

Erasmus Mundus MSc. Mathematical Modeling

Outline

Introduction

Time series are sequences of data points, typically consisting of successive measurements made over a time interval.

Time series are used in

- statistics, econometrics, weather forecasting
- mathematical finance and in any domain of applied science involving temporal measurements.

In this lecture we will discuss the Basics of Time Series for AR(1), MA(1), ARMA (1,1)

Stochastic Process

A stochastic process

$$\{Y_1, Y_2, \ldots Y_t, \ldots\} = \{Y_t\}_{t=1}^{\infty}$$

is a sequence of random variables indexed by time.

Stochastic Process (II)

A Realization of a stochastic process up to time T

$$\{Y_1 = y_1, Y_2 = y_2, ...Y_T = y_T\} = \{y_t\}_{t=1}^{T}$$

is a sequence of data points y_i indexed by time.

Stochastic Process

There are two important forms of stationarity:

- strictly (or strong) stationarity
- covariance (or weak) stationarity

Strictly Stationary Process

A stochastic process is strictly stationary if for any set $t_1, t_2, t_3, .., t_r$ the joint probability distribution of $\{Y_{t_1} Y_{t_2}, ... Y_{t_r}\}$does not change when shifted in time.

Strictly stationary example

$$\{Y_1, Y_3, Y_{10}\} \sim \{Y_5, Y_7, Y_{14}\}$$

The Joint Distribution is Time shift Invariant!

Covariance Stationary Process

A stochastic process is **Covariance Stationary** if

$$E\left[Y_t\right] = \mu \text{ ind } t$$

$$var\left(Y_t\right) = \sigma^2 \text{ ind } t$$

$$cov(Y_t, Y_{t-j}) = \gamma_j \text{ depends } j$$

the covariance between any two terms of the sequence depends only on the relative positions j of the two terms

Covariance Stationary Process II

The covariance $cov(Y_t, Y_{t-j}) = \gamma_j$ is a measure of the **linear** dependence direction for Y_t, Y_{t-j}. And The Correlation ρ_j defined as

$$\rho_j = \frac{cov(Y_t, Y_{t-j})}{\sqrt{var(Y_t) var(Y_{t-j})}} = \frac{\gamma_j}{\sigma^2}$$

measures its strenght and it satisfies

$$-1 \leq \rho_j \leq 1$$

Gaussian White Noise

A process$\{Y_t\}_{t=1}^{\infty}$ where $Y_t \sim iid\ N\left(0, \sigma^2\right)$ is called Gaussian White Noise *GWN* process and satisfies

$$E\left[Y_t\right] = 0 \text{ ind } t$$

$$var\left(Y_t\right) = \sigma^2 \text{ ind } t$$

$$cov(Y_t, Y_{t-j}) = 0 \text{ for all } j > 0 \text{ ind } t$$

other types are *Weak-WN* (not ind but just uncorr.) and *Independent-WN* (not gaussian but any distr.)

Gaussian White Noise (II)

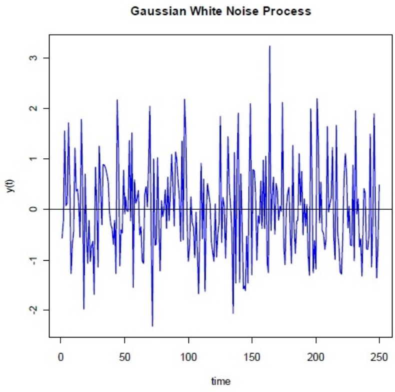

Figure: Gaussian White Noise $GWN(0, 1)$

Non- Stationary Process

In a Non- Stationary Process maybe the mean, the autocovariance or both might depend on$t$. For Example,

- Trending Process
- Random Walk

The Trending Process

The deterministic trending process is

$$Y_t = \beta_0 + \beta_1 t + \varepsilon_t$$

where

$$\varepsilon_t \sim iid\ N\left(0, \sigma_\varepsilon^2\right)$$

$$E\left[Y_t\right] = \beta_0 + \beta_1 t$$

$$var\left(Y_t\right) = E\left[\varepsilon_t\right]^2 = \sigma_\varepsilon^2$$

the mean is a function of t.

The Trending Process (II)

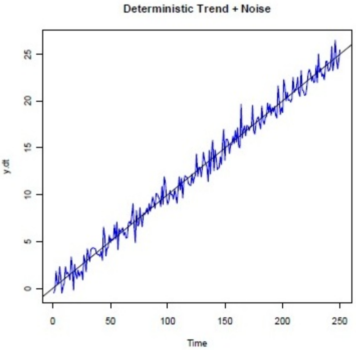

Figure: Deterministic trending $Y_t = 0.1 * t + \epsilon_t$, $\epsilon_t \sim N(0, 1)$

Random Walk

The Random Walk is defined as

$$Y_t = Y_{t-1} + \varepsilon_t$$

where

$$\varepsilon_t \sim iid\ N\left(0, \sigma_\varepsilon^2\right)$$

by recursive substitution

$$Y_t = Y_0 + \sum_{i=1}^{t} \varepsilon_i$$

Random Walk (II)

Random Walk

$$E\left[Y_t\right] = Y_0$$

$$var\left(Y_t\right) = t \cdot \sigma_{\varepsilon}^{2}$$

the variance increase linearly as function of t.

Random Walk (III)

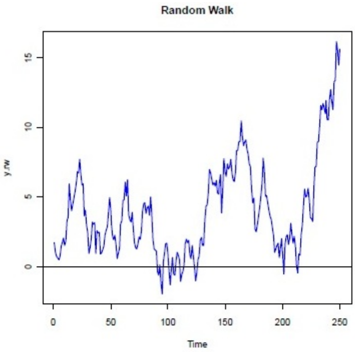

Figure: Random walk $Y_t = Y_{t-1} + \epsilon_t$, $\epsilon_t \sim N(0, 1)$

Example (II)

Let's consider

$$Y_t \sim GWN\left(0,1\right) \;\; X \sim N\left(0,1\right) \; s.t \; X \perp Y_t$$

then $Z_t = Y_t + X$ implies that

$$var\left(Z_t\right) = 2 \text{ and } cov\left(Z_t, Z_{t-j}\right) = 1$$
$$\Rightarrow \rho_j = \frac{cov(Y_t, Y_{t-j})}{\sqrt{var(Y_t)var(Y_{t-j})}} = \frac{1}{2}$$

for all j. The correlation doesn't vanish with time!

Moving Average Processes

MA(1) is a process in which the correlation last 1 time period

$$Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$$

$$\varepsilon_t \sim iid\ N\left(0, \sigma_\varepsilon^2\right)$$

Moving Average Processes (II)

MA(1) satisfy

$$E\left[Y_t\right] = \mu$$

$$var\left(Y_t\right) = \sigma_{\varepsilon}^2\left(1+\theta^2\right)$$

$$cov(Y_t, Y_{t-1}) = \theta\sigma_{\varepsilon}^2$$

Moving Average Processes (III)

$$corr(Y_t, Y_{t-1}) = \frac{\theta}{(1+\theta^2)}$$

we can observe that

$$\begin{cases} \theta = 0 & \rho = 0 \\ \theta > 0 & \rho > 0 \\ \theta < 0 & \rho < 0 \\ |\theta| = 1 & |\rho_{MAX}| = 0.5 \end{cases}$$

and that for any j bigger than 1

$$corr(Y_t, Y_{t-j}) = 0$$

Moving Average Processes (IV)

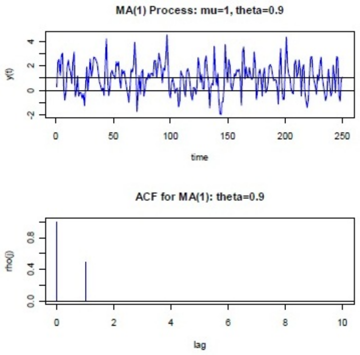

Figure: MA(1) $\mu = 1,\ \theta = 0.9\ \sigma_{\epsilon}^{2} = 1$

AutoRegressive Processes

AR(1) Model are

$$Y_t = \mu + \phi\,(Y_{t-1} - \mu) + \varepsilon_t$$

$$\varepsilon_t \sim iid\ N\left(0, \sigma_\varepsilon^2\right)$$

now the correlation decays to zero progressively if

$$-1 < \phi < 1$$

AutoRegressive Processes (II)

For an AR(1) Model we have

$$E\left[Y_t\right] = \mu$$

$$var\left(Y_t\right) = \sigma_\varepsilon^2/(1-\phi^2)$$

$$cov(Y_t, Y_{t-1}) = \frac{\sigma_\varepsilon^2 \phi}{(1-\phi^2)}$$

$$corr(Y_t, Y_{t-1}) = \phi$$

AutoRegressive Processes (III)

for any j bigger than 1

$$cov(Y_t, Y_{t-j}) = \frac{\sigma_\varepsilon^2}{(1-\phi^2)}\phi^j$$

$$corr(Y_t, Y_{t-j}) = \phi^j$$

$$lim_{j\to\infty}\, \phi^j = 0$$

the closer is ϕ to unity the stronger the correlation in time

AutoRegressive Processes (IV)

it can be written in the linear regression form (useful for estimation using least squares)

$$Y_t = c + \phi Y_{t-1} + \varepsilon_t$$

where

$$c = \mu (1 - \phi)$$

AutoRegressive Processes (V)

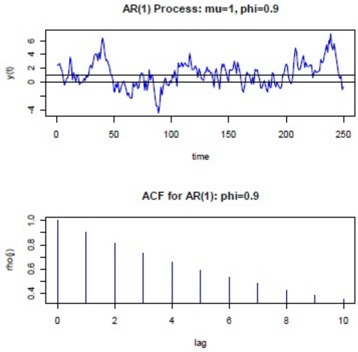

Figure: AR(1) $\mu = 1$, $\theta = 0.9$ $\sigma_\epsilon^2 = 1$

ARMA(1,1)

An ARMA(1,1) process (sum of MA(1) and AR(1)) is written as

$$Y_t = c + \varepsilon_t + \phi Y_{t-1} + \theta \varepsilon_{t-1}$$

see that if $\phi = 0 \Rightarrow MA(1)$ and if $\theta = 0 \Rightarrow AR(1)$. Then

$$E[Y_t] = c + \phi E[Y_{t-1}]$$

and

$$E[Y_t] = \frac{c}{1-\phi}$$

ARMA(1,1) II

Now the variance is

$$var\left(Y_t\right) = \frac{\left(1+\theta^2+2\phi\theta\right)}{\left(1-\phi^2\right)}\sigma_\epsilon^2$$

and

$$cov(Y_t, Y_{t-1}) = \phi var\left(Y_{t-1}\right) + \theta\sigma_\epsilon^2 = \frac{\left(\phi+\theta\right)\left(1+\phi\theta\right)}{\left(1-\phi^2\right)}\sigma_\epsilon^2$$

Conclusions

We have studied the basic properties of Time Series Processes

- MA(1)
- AR(1)
- ARMA(1,1)

References

E. Zivot. Time Series Concepts. *Manuscript for Introduction to Computational Finance*. 2012