Calibration of the Vasicek Model: An Step by Step Guide

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Abstract

In this report we present 3 methods for calibrating the Ornstein–Uhlenbeck process to a data set. The model is described and the sensitivity analysis with respect to changes in the parameters is performed. In particular the Least Squares Method, the Maximum Likelihood Method and the Long Term Quantile Method are presented in detail.

Introduction

The Ornstein–Uhlenbeck process [3] (named after Leonard Ornstein and George Eugene Uhlenbeck), is a stochastic process that, over time, tends to drift towards its long-term mean: such a process is called mean-reverting. It can also be considered as the continuous-time analogue of the discrete-time AR(1) process where there is a tendency of the walk to move back towards a central location, with a greater attraction when the process is further away from the center.
Vasicek assumed that the instantaneous spot Interest Rate under the real world measure evolves as an Orstein-Uhlenbeck process with constant coefficients [5]. The most important feature which this model exhibits is the mean reversion,which means that if the interest rate is bigger than the long run mean μ, then the coefficient λ makes the drift become negative so that the rate will be pulled down in the direction of μ. Similarly, if the interest rate is smaller than the long run mean. Therefore, the coefficient λ is the speed of adjustment of the interest rate towards its long run level. This model is of particular interest in finance because there are also compelling economic arguments in favor of mean reversion. When the rates are high, the economy tends to slow down and borrowers require less funds. Furthermore, the rates pull back to its equilibrium value and the rates decline. On the contrary when the rates are low, there tends to be high demand for funds on the part of the borrowers and rates tend to increase. One unfortunate consequence of a normally distributed interest rate is that it is possible for the interest rate to become negative.
In this article we start from the Euler Maruyana discretization scheme for the Vasicek process and the sensitivity analysis, then we present in detail 3 well known methods (Least squares Method, Maximum Likelihood Method and the Long Term Quantile Method) for calibrating the model's parameter to a data set. We also refer the reader to [4] where some of these techniques are applied but using a different scheme.

1 Euler Scheme and Sensitivity Analysis

The stochastic differential equation (SDE) for the Ornstein-Uhlenbeck process is given by

$$dr_t = \lambda\left(\mu - r_t\right)dt + \sigma dW_t \tag{1.1}$$

with λ the mean reversion rate, μ the mean, and σ the volatility. The solution of the model is

$$r_t = r_0 \exp\left(-\lambda t\right) + \mu\left(1 - \exp\left(-\lambda t\right)\right) + \sigma \int_0^t \exp\left(-\lambda t\right) dW_t \tag{1.2}$$

Here the interest rates are normally distributed and the expectation and variance are given by

$$E_o\left[r_t\right] = r_0 \exp\left(-\lambda t\right) + \mu\left(1 - \exp\left(-\lambda t\right)\right) \tag{1.3}$$

and

$$Var\left[r_t\right] = \frac{\sigma^2}{2\lambda}\left(1 - \exp\left(-2\lambda t\right)\right) \tag{1.4}$$

as $t \to \infty$, the limit of expected rate and variance, will converge to μ and $\frac{\sigma^2}{2\lambda}$ respectively. The Euler Maruyana Scheme for this models is

$$r_{t+\delta t} = r_t + \lambda\left(\mu - r_t\right)\delta t + \sigma\sqrt{\delta t}\mathcal{N}\left(0,1\right) \tag{1.5}$$

the process can go negative with probability

$$P\left(r_{t+\delta t} \leq 0\right) = P\left(r_t + \lambda\left(\mu - r_t\right)\delta t + \sigma\sqrt{\delta t}\mathcal{N}\left(0,1\right) \leq 0\right) \tag{1.6}$$

$$= P\left(\mathcal{N}\left(0,1\right) \leq -\left(\frac{r_t + \lambda\left(\mu - r_t\right)\delta t}{\sigma\sqrt{\delta t}}\right)\right) \tag{1.7}$$

$$= \Phi\left(-\frac{r_t + \lambda\left(\mu - r_t\right)\delta t}{\sigma\sqrt{\delta t}}\right) \tag{1.8}$$

Some of the parameters play a big role in the pricing of a financial derivatives or in the forecasting of a process, while some of them do not affect them so much. Therefore, depending on what we want to price or forecast, it is important to check the sensitivity of the models with respect to different parameters. The corresponding sensitivity analysis is performed as presented in [6]. Lets consider a two outcomes of a process which differ exclusively in a perturbation of one of the parameters (but they have the same stochastic realization $\mathcal{N}\left(0,1\right)$), then for λ we have that

$$\tilde{r}_{t+\delta t} = r_t + \left[\lambda + \Delta\lambda\right]\left(\mu - r_t\right)\delta t + \sigma\sqrt{\delta t}\mathcal{N}\left(0,1\right) \tag{1.9}$$

so

$$\tilde{r}_{t+\delta t} - r_{t+\delta t} = \Delta\lambda\left(\mu - r_t\right)\delta t \tag{1.10}$$

when λ is increased the variance (1.4) decreases. So, the change in the reversion coefficient will not affect the short rate (1.3) in long term, just effect the time which is necessary for the interest rate to come back to the long term mean. Therefore, λ is important in the pricing of the financial instruments which are affected by the volatility (1.4), but are not dependent on the long term expected value (1.3) of the simulated interest rate. For μ is performed as

$$\tilde{r}_{t+\delta t} = r_t + \lambda\left(\left[\mu + \Delta\mu\right] - r_t\right)\delta t + \sigma\sqrt{\delta t}\mathcal{N}\left(0,1\right) \tag{1.11}$$

so

$$\tilde{r}_{t+\delta t} - r_{t+\delta t} = \lambda\left(\Delta\mu - r_t\right)\delta t \tag{1.12}$$

for σ is performed as

$$\tilde{r}_{t+\delta t} = r_t + \lambda\left(\mu - r_t\right)\delta t + \left[\sigma + \Delta\sigma\right]\sqrt{\delta t}\mathcal{N}\left(0,1\right) \tag{1.13}$$

so

$$\tilde{r}_{t+\delta t} - r_{t+\delta t} = \Delta\sigma\sqrt{\delta t}\mathcal{N}\left(0,1\right) \tag{1.14}$$

Because of the standard Brownian motion, in the long term, the effect of the change in σ does not affect the expected value of the interest rate (1.3), but it increases the variance (1.4).

```
%% Euler Scheme Vasicek
clc
clear all
close all

%% Set the seed at 123
```

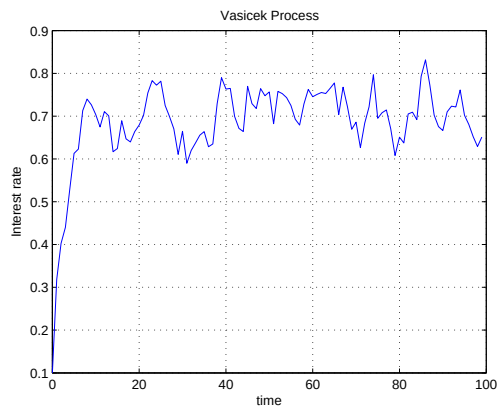

Figure 1.1: Vasicek Process

```
rng(123)

%% Parameters of the Model
lambda=0.3;% Revertion coefficient
N=3; % Number of simulations
mu=0.7; % Long term Mean
sigma=0.05; % Volatility
delta_t=1; % Time step
T=100; % Time lenght
color=['b','r','m'];% color
n=T./delta_t;% Number of time steps
j=1;

%% Simulating Vasicek Euler Scheme
S0=0.1; % Starting point
i=2;
S(1)=S0; % Starting point
while i<n+1
S(i)= S(i-1) + lambda.*(mu-S(i-1))*delta_t + sigma.*(sqrt(delta_t).*randn(1));
i=i+1;
end
```

2 Least Squares Calibration

The idea of least squares is that we choose parameter estimates that minimize the average squared difference between observed and predicted values. That is, we maximize the fit of the model to the data by choosing the model that is closest, on average, to the data. Rewriting (1.5) we have

$$r_{t+\delta t} = r_t\left(1-\lambda\delta t\right) + \lambda\mu\delta t + \sigma\sqrt{\delta t}\mathcal{N}\left(0,1\right) \tag{2.1}$$

The relationship between consecutive observations $r_{t+\delta t}$ and r_t is linear with a *iid* normal random term ϵ

$$r_{t+\delta t} = ar_t + b + \epsilon \tag{2.2}$$

or

$$\left[r_{t+\delta t}\right] = \left[\begin{array}{cc} r_t & 1 \end{array}\right]\left[\begin{array}{c} a \\ b \end{array}\right] + \epsilon \tag{2.3}$$

where

$$a = 1-\lambda\delta t \tag{2.4}$$

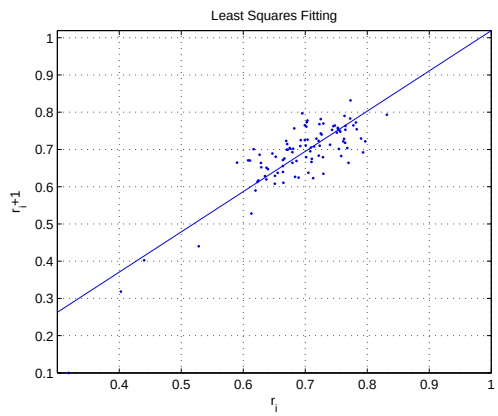

Figure 2.1: Least Squares Fitting

$$b = \lambda \mu \delta t \tag{2.5}$$

$$\epsilon = \sigma \sqrt{\delta t} \mathcal{N}(0,1) \tag{2.6}$$

where using a least squares fitting as described in the Appendix.

$$\hat{a} = \frac{\sum_{i=1}^{n} (r_{i+1} r_i) - \frac{1}{n} \left(\sum_{i=1}^{n} r_i \sum_{i=1}^{n} r_{i+1} \right)}{\sum_{i=1}^{n} r_i^2 - \frac{1}{n} \left(\sum_{i=1}^{n} r_i \sum_{i=1}^{n} r_i \right)} \tag{2.7}$$

and

$$\hat{b} = \frac{1}{n} \left(\sum_{i=1}^{n} r_{i+1} - \sum_{i=1}^{n} a r_i \right) \tag{2.8}$$

so we can estimate the model's parameters as

$$\boxed{\lambda = \frac{1-a}{\delta t}} \tag{2.9}$$

$$\boxed{\mu = \frac{b}{1-a}} \tag{2.10}$$

$$\boxed{\sigma^2 = \frac{Var(\epsilon)}{\delta t}} \tag{2.11}$$

we refer the reader to the Appendix (4) where the derivation and the implemented code is presented.

3 Maximum Likelihood Calibration

In maximum likelihood estimation, we search over all possible sets of parameter values for a specified model to find the set of values for which the observed sample was most likely. That is, we find the set of parameter values that, given a model, were most likely to have given us the data that we have in hand. The distribution of $r_{t+\delta t}$ in the Euler scheme is given by

$$f(r_{t+\delta t} \mid r_t, \mu, \lambda, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2 \delta t}} \exp - \frac{\left[r_{t+\delta t} - (r_t + \lambda(\mu - r_t)\delta t) \right]^2}{2\sigma^2 \delta t} \tag{3.1}$$

so the log-likelihood

$$\mathcal{L} = \ln \prod_{i=1}^{n} f(r_i \mid r_{i-1}, \mu, \lambda, \sigma) = \sum_{i=1}^{n} \ln f(r_i \mid r_{i-1}, \mu, \lambda, \sigma) \tag{3.2}$$

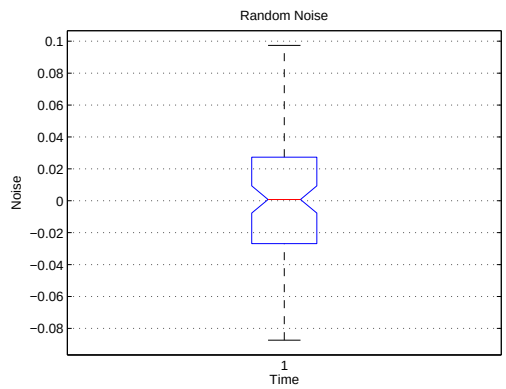

Figure 3.1: Noise Box Plot

$$=\sum_{i=1}^{n} \ln\left[\frac{1}{\sqrt{2\pi\sigma^2\delta t}} \exp - \frac{\left[r_i - \left(r_{i-1} + \lambda\left(\mu - r_{i-1}\right)\delta t\right)\right]^2}{2\sigma^2\delta t}\right] \tag{3.3}$$

$$=\sum_{i=1}^{n} \ln\left[\frac{1}{\sqrt{2\pi\sigma^2\delta t}}\right] - \frac{\left[r_i - \left(r_{i-1} + \lambda\left(\mu - r_{i-1}\right)\delta t\right)\right]^2}{2\sigma^2\delta t} \tag{3.4}$$

$$= \ln\left[\sqrt{2\pi\sigma^2\delta t}\right]^{-n} - \sum_{i=1}^{n} \frac{\left[r_i - \left(r_{i-1} + \lambda\left(\mu - r_{i-1}\right)\delta t\right)\right]^2}{2\sigma^2\delta t} \tag{3.5}$$

obtaining

$$\boxed{\mathcal{L} = -\frac{n}{2}\ln\left[2\pi\sigma^2\delta t\right] - \sum_{i=1}^{n} \frac{\left[r_i - \left(r_{i-1} + \lambda\left(\mu - r_{i-1}\right)\delta t\right)\right]^2}{2\sigma^2\delta t}} \tag{3.6}$$

using the procedure described in the Appendix (4) the parameters are estimated giving

$$\boxed{\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{r_i - r_{i-1}\left(1 - \lambda\delta t\right)}{\lambda\delta t}\right]} \tag{3.7}$$

$$\boxed{\lambda = \frac{1}{\delta t}\frac{\sum_{i=1}^{n}\left(r_i - r_{i-1}\right)\left(\mu - r_{i-1}\right)}{\sum_{i=1}^{n}\left(\mu - r_{i-1}\right)^2}} \tag{3.8}$$

$$\boxed{\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}\frac{\left[r_i - \left(r_{i-1} + \lambda\left(\mu - r_{i-1}\right)\delta t\right)\right]^2}{\delta t}} \tag{3.9}$$

we refer the reader to the Appendix (4) where the derivation and the implemented code is presented.

```
% Y-B(1).X-B(2)=    Noise
de_trended=Y-B(1).*X-B(2)

figure(5)
histfit(de_trended,13)
figure(6)
probplot('normal',de_trended)
grid on
```

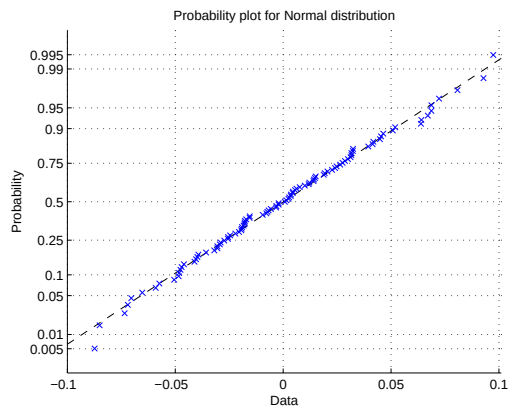

Figure 3.2: Normal Plot

4 Long Term Quantile Method

The major assumption in this model is that the quantiles from the historical data are representative for quantiles in the future. Therefore, a 95% confidence interval is taken from the historical data and the parameters in the short interest rate model are chosen such that in 95% of the cases the generated interest rates will fall within the confidence interval taken from the historical data.

$$r_t \sim \mathcal{N}\left(r_0 \exp\left(-\lambda t\right) + \mu\left(1 - \exp\left(-\lambda t\right)\right), \frac{\sigma^2}{2\lambda}\left(1 - \exp\left(-2\lambda t\right)\right)\right) \tag{4.1}$$

The difference between the long term standard deviation and the deviation at t is

$$\Delta\sigma = \sqrt{\frac{\sigma^2}{2\lambda}\left(1 - \exp\left(-2\lambda t\right)\right)} - \sqrt{\frac{\sigma^2}{2\lambda}} \tag{4.2}$$

using Taylor expansion for $\exp\left(-2\lambda t\right) \lll 1$

$$\Delta\sigma = \sqrt{\frac{\sigma^2}{2\lambda}}\left[\left(1 + \frac{1}{2}\exp\left(-2\lambda t\right)\right) - 1\right] \tag{4.3}$$

so

$$\frac{\Delta\sigma}{\sigma} = \frac{1}{2}\exp\left(-2\lambda t\right) \ll 1 \tag{4.4}$$

the difference between the long term mean and the mean at t is

$$\Delta\mu = \left(r_0 - \mu\right)\exp\left(-\lambda t\right) \tag{4.5}$$

so

$$\frac{\Delta\mu}{\mu} = \left(\frac{r_0}{\mu} - 1\right)\exp\left(-\lambda t\right) \tag{4.6}$$

The confidence interval is the one such that

$$P\left(\mu - 1.96\frac{\sigma}{\sqrt{2\lambda}} \leq \lim_{t\to\infty} r_t \leq \mu + 1.96\frac{\sigma}{\sqrt{2\lambda}}\right) = 0.95 \tag{4.7}$$

calling

$$\tilde{q}_{0.95} = \mu + 1.96\frac{\sigma}{\sqrt{2\lambda}} \tag{4.8}$$

and

$$\tilde{q}_{0.05} = \mu - 1.96\frac{\sigma}{\sqrt{2\lambda}} \tag{4.9}$$

we can obtain the parameters as

$$\boxed{\mu = \frac{\tilde{q}_{0.95} + \tilde{q}_{0.05}}{2}} \tag{4.10}$$

and

$$\boxed{\lambda = 2\frac{\sigma^2 (1.96)^2}{(\tilde{q}_{0.95} - \tilde{q}_{0.05})^2}} \tag{4.11}$$

```
%% Montecarlo Simulation
j=1;

while j<10000
S(1)=S0; % Starting point

while i<n+1
S(i)= S(i-1) + lambda.*(mu-S(i-1))*delta_t + sigma.*(sqrt(delta_t).*randn(1));
i=i+1;
end

MC(j)=S(T);
j=j+1;
end

figure(9)
h=histfit(MC,12)
title('Distribution of the Vasicek process by Montecarlo Simulation') grid on
```

the error associated with λ is

$$\ln \lambda = \ln 2 (1.96)^2 + \ln \sigma^2 - \ln (x - y)^2 \tag{4.12}$$

using partial differentiation we have that the maximum percentual error (for a detailed discussion on errors analysis see [1, 2]) is given by

$$\frac{\triangle \lambda}{\lambda} = 2 \left| \frac{\triangle \sigma}{\sigma} \right| + 2 \left| \frac{(\triangle \tilde{q}_{0.95} + \triangle \tilde{q}_{0.05})}{(\tilde{q}_{0.95} - \tilde{q}_{0.05})} \right| \tag{4.13}$$

```
z=quantile(MC,0.95)  y=quantile(MC,0.05)
% Mu
(z+y)./2
% Lambda
2*(1.96.*sigma/(z-y)).^2
```

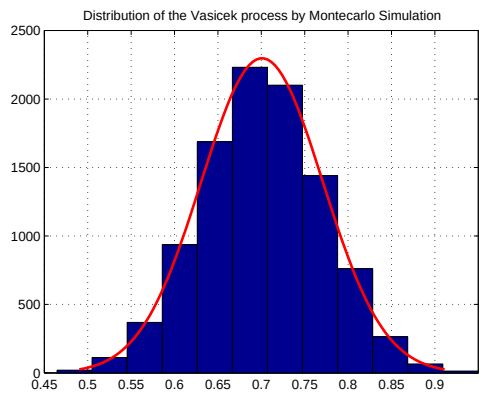

Figure 4.1: Montecarlo Histogram

Appendix

Least Squares Fitting

The residuals for the model are given by

$$R_i = r_{i+1} - (ar_i + b) \tag{4.14}$$

This method minimizes the sum of squared residuals, which is given by

$$S = \sum_{i=1}^{n} R_i^2; = \sum_{i=1}^{n} (r_{i+1})^2 + \sum_{i=1}^{n} (ar_i + b)^2 - 2\sum_{i=1}^{n} (r_{i+1}(ar_i + b)) \tag{4.15}$$

The least squares estimators for the parameters can then be found by differentiating S with respect to these parameters and setting these derivatives equal to zero. For b we have that

$$\frac{\partial S}{\partial b} = 2\sum_{i=1}^{n} (ar_i + b) - 2\sum_{i=1}^{n} r_{i+1} \tag{4.16}$$

$$\sum_{i=1}^{n} ar_i + nb = \sum_{i=1}^{n} r_{i+1} \tag{4.17}$$

$$\boxed{b = \frac{1}{n}\left(\sum_{i=1}^{n} r_{i+1} - \sum_{i=1}^{n} ar_i\right)} \tag{4.18}$$

For a we have that

$$\frac{\partial S}{\partial a} = 2\sum_{i=1}^{n} (ar_i + b)\, r_i - 2\sum_{i=1}^{n} (r_{i+1} r_i) \tag{4.19}$$

isolating

$$\sum_{i=1}^{n} (ar_i + b)\, r_i = \sum_{i=1}^{n} (r_{i+1} r_i) \tag{4.20}$$

then

$$\sum_{i=1}^{n} ar_i^2 + \sum_{i=1}^{n} br_i = \sum_{i=1}^{n} (r_{i+1} r_i) \tag{4.21}$$

using 4.18 we have

$$\sum_{i=1}^{n} ar_i^2 + \frac{1}{n}\sum_{i=1}^{n} r_i \left(\sum_{i=1}^{n} r_{i+1} - \sum_{i=1}^{n} ar_i\right) = \sum_{i=1}^{n} (r_{i+1} r_i) \tag{4.22}$$

is equal to

$$\sum_{i=1}^{n} a r_i^2 - \frac{1}{n} a \left(\sum_{i=1}^{n} r_i \sum_{i=1}^{n} r_i \right) + \frac{1}{n} \left(\sum_{i=1}^{n} r_i \sum_{i=1}^{n} r_{i+1} \right) = \sum_{i=1}^{n} (r_{i+1} r_i) \tag{4.23}$$

grouping

$$a \left[\sum_{i=1}^{n} r_i^2 - \frac{1}{n} \left(\sum_{i=1}^{n} r_i \sum_{i=1}^{n} r_i \right) \right] = \sum_{i=1}^{n} (r_{i+1} r_i) - \frac{1}{n} \left(\sum_{i=1}^{n} r_i \sum_{i=1}^{n} r_{i+1} \right) \tag{4.24}$$

finally

$$\boxed{a = \frac{\sum_{i=1}^{n} (r_{i+1} r_i) - \frac{1}{n} \left(\sum_{i=1}^{n} r_i \sum_{i=1}^{n} r_{i+1} \right)}{\sum_{i=1}^{n} r_i^2 - \frac{1}{n} \left(\sum_{i=1}^{n} r_i \sum_{i=1}^{n} r_i \right)}} \tag{4.25}$$

```
%% Calibration using Least Squares regression

% Plot S_i vs S_i-1
figure(3)
Y= S(2:end);% removing first point
X=S(1:end-1);% removing the last point
plot(Y,X,'.')
lsline % least squares line
ylabel('r_i+1') xlabel('r_i')
title('Least Squares Fitting')
grid on

%Rewrite the offset term
offset=ones(size(S,2)-1,1);
new_X=[X',offset];
B = new_X\Y' % Solves in the Least Squares sense

est_lambda=1-B(1)./delta_t
est_mu= B(2)./(1-B(1))
```

Maximum Likelihood Fitting

An Estimator for μ

$$\frac{\partial \mathcal{L}}{\partial \mu} = \sum_{i=1}^{n} - \frac{2 \left[r_i - \left(r_{i-1} + \lambda \left(\mu - r_{i-1} \right) \delta t \right) \right] \lambda \mu \delta t}{2 \sigma^2 \delta t} \tag{4.26}$$

$$= \sum_{i=1}^{n} - \frac{\left[r_i - \left(r_{i-1} \left(1 - \lambda \delta t \right) + \lambda \mu \delta t \right) \right] \lambda \mu}{\sigma^2} = 0 \tag{4.27}$$

$$\sum_{i=1}^{n} \left[r_i - \left(r_{i-1} \left(1 - \lambda \delta t \right) + \lambda \mu \delta t \right) \right] = 0 \tag{4.28}$$

$$\sum_{i=1}^{n} \lambda \mu \delta t = \sum_{i=1}^{n} \left[r_i - r_{i-1} \left(1 - \lambda \delta t \right) \right] \tag{4.29}$$

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \left[\frac{r_i - r_{i-1} \left(1 - \lambda \delta t \right)}{\lambda \delta t} \right] \tag{4.30}$$

An Estimator for

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{i=1}^{n} -\frac{\left[r_i - \left(r_{i-1} + \lambda\left(\mu - r_{i-1}\right)\delta t\right)\right]\left(\mu - r_{i-1}\right)\delta t}{\sigma^2 \delta t} \tag{4.31}$$

$$= \sum_{i=1}^{n} -\frac{\left[r_i - \left(r_{i-1} + \lambda\left(\mu - r_{i-1}\right)\delta t\right)\right]\left(\mu - r_{i-1}\right)}{\sigma^2} \tag{4.32}$$

$$= \sum_{i=1}^{n} -\frac{\left[\left(r_i - r_{i-1}\right)\left(\mu - r_{i-1}\right) - \lambda\left(\mu - r_{i-1}\right)^2 \delta t\right]}{\sigma^2} = 0 \tag{4.33}$$

$$\sum_{i=1}^{n} \left(r_i - r_{i-1}\right)\left(\mu - r_{i-1}\right) = \sum_{i=1}^{n} \lambda\left(\mu - r_{i-1}\right)^2 \delta t \tag{4.34}$$

$$\lambda = \frac{1}{\delta t} \frac{\sum_{i=1}^{n} \left(r_i - r_{i-1}\right)\left(\mu - r_{i-1}\right)}{\sum_{i=1}^{n} \left(\mu - r_{i-1}\right)^2} \tag{4.35}$$

An estimator for σ

$$\frac{\partial \mathcal{L}}{\partial \sigma} = -\frac{n}{4\pi\sigma^2\delta t}\left(4\pi\sigma\delta t\right) - \sum_{i=1}^{n} -2\frac{\left[r_i - \left(r_{i-1} + \lambda\left(\mu - r_{i-1}\right)\delta t\right)\right]^2}{2\sigma^3\delta t} \tag{4.36}$$

$$= -\frac{n}{\sigma} + \sum_{i=1}^{n} \frac{\left[r_i - \left(r_{i-1} + \lambda\left(\mu - r_{i-1}\right)\delta t\right)\right]^2}{\sigma^3\delta t} = 0 \tag{4.37}$$

$$\frac{n}{\sigma} = \sum_{i=1}^{n} \frac{\left[r_i - \left(r_{i-1} + \lambda\left(\mu - r_{i-1}\right)\delta t\right)\right]^2}{\sigma^3\delta t} \tag{4.38}$$

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} \frac{\left[r_i - \left(r_{i-1} + \lambda\left(\mu - r_{i-1}\right)\delta t\right)\right]^2}{\delta t} \tag{4.39}$$

```
%% Calibration using Maximum Likelihood Estimators
n = length(S)-1;
Sx  = sum( S(1:end-1) );
Sy  = sum( S(2:end) );
Sxx = sum( S(1:end-1).^2 );
Sxy = sum( S(1:end-1).*S(2:end) );
Syy = sum( S(2:end).^2 );

mu  = (Sy*Sxx - Sx*Sxy) / ( n*(Sxx - Sxy) - (Sx^2 - Sx*Sy) );
lambda = ( (Sxy - mu*Sx - mu*Sy + n*mu^2) / (Sxx -2*mu*Sx + n*mu^2) ) / delta_t;
a = 1-lambda*delta_t;
sigmah2 = (Syy - 2*a*Sxy + a^2*Sxx - 2*mu*(1-a)*(Sy - a*Sx) + n*mu^2*(1-a)^2)/n;
sigma = sqrt(sigmah2*2*lambda/(1-a^2));
end
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